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CERTAIN GROUPS OF MOTIONS IN 3-SPACE OF CONSTANT CURVATURE

By A. G. WALKER (*Liverpool*)

[Received 28 December 1939]

1. Introduction. The complete group G_6 of motions in a Riemannian 3-space S_3 of constant curvature has been considered by H. P. Robertson,† who also discussed certain sub-groups of G_6. His results, when applied to space with positive definite metric, show that there are only two invariant transitive three-parameter sub-groups, these being reciprocal, and that they are real only when the curvature of S_3 is positive, excluding the trivial case when the curvature is zero. A transitive three-parameter sub-group occurs in relativistic cosmology,‡ and, since it has not been possible up to the present to discard one sign of curvature in favour of the other, it is of some interest to consider sub-groups when the curvature is negative, although they are not then invariant. The implications of non-invariance will be discussed later.

We shall proceed therefore to find all the real transitive three-parameter sub-groups of G_6. Standard notation will be used as far as possible, and known results in Riemannian geometry§ and group theory‖ will be quoted without detailed explanation. A summation convention will be used, where a repeated suffix implies summation regardless of the positions of the suffixes: thus $a_\alpha b_\alpha$, $a_\alpha b^\alpha$, $a^\alpha b^\alpha$ all denote sums. All unprimed suffixes take values 1, 2, 3 except θ, ϕ, ψ which refer to G_6 and take values 1, 2,..., 6. Primed suffixes take values 4, 5, 6, and are related to the unprimed suffixes thus: $\lambda' = \lambda+3$; the summation convention applies regardless of the primes, thus $a_\alpha b^{\alpha'}$ denotes $a_1 b^4+a_2 b^5+a_3 b^6$.

2. Coordinates x^i can be chosen so that the metric, taken to be positive definite, of a Riemannian 3-space S_3 of constant curvature K becomes

$$ds^2 = \frac{dx^i dx^i}{(1+\frac{1}{4}Kr^2)^2} \quad (r^2 = x^i x^i). \tag{1}$$

If $K > 0$, the x's can take all values, and, if $K < 0$, the x's must

† *Annals of Math.* (2), 33 (1932), 496–521.

‡ A. G. Walker, *Proc. London Math. Soc.* (2), 46 (1940), 113–54.

§ See Eisenhart, *Riemannian Geometry* (Princeton, 1926).

‖ See Eisenhart, *Continuous Groups of Transformations* (Princeton, 1933).

satisfy $r^2 < -4/K$. With these coordinates Robertson has shown† that the vectors of the group G_6 of motions in S_3 can be taken to be

$$\begin{aligned} \eta^i_\alpha &= (1-\tfrac{1}{4}Kr^2)\delta^i_\alpha + \tfrac{1}{2}Kx^i x^\alpha \quad (\alpha = 1, 2, 3), \\ \eta^i_{\alpha'} &= e_{i\alpha k}x^k \qquad\qquad\qquad (\alpha' = 4, 5, 6), \end{aligned} \tag{2}$$

where e_{ijk} is zero if two suffixes are equal and is ± 1 according as (ijk) is an even or odd permutation of (123). The constants of structure of G_6 are $\gamma^\theta_{\phi\psi}$, where

$$\eta^k_\phi \eta^i_{\psi.k} - \eta^k_\psi \eta^i_{\phi.k} = \gamma^\theta_{\phi\psi}\eta^i_\theta \quad (\phi, \psi = 1, 2, \ldots, 6), \tag{3}$$

a point denoting partial differentiation. Substituting from (2), we find

$$\begin{aligned} \gamma^\epsilon_{\alpha\beta} &= 0, & \gamma^{\epsilon'}_{\alpha\beta} &= -Ke_{\epsilon\alpha\beta}, \\ \gamma^\epsilon_{\alpha\beta'} &= -e_{\epsilon\alpha\beta}, & \gamma^{\epsilon'}_{\alpha\beta'} &= 0, \\ \gamma^\epsilon_{\alpha'\beta'} &= 0, & \gamma^{\epsilon'}_{\alpha'\beta'} &= -e_{\epsilon\alpha\beta}, \end{aligned} \tag{4}$$

the remaining constants being given by $\gamma^\theta_{\phi\psi} + \gamma^\theta_{\psi\phi} = 0$.

Each vector η^i_θ generates a one-parameter sub-group of G_6 and it is easily verified that a vector η^i_α ($\alpha = 1, 2, 3$) generates motions in S_3 equivalent to translations, and a vector $\eta^i_{\alpha'}$ ($\alpha' = 4, 5, 6$) generates rotations about the origin $x^i = 0$. The three vectors $\eta^i_{\alpha'}$ are, in fact, the vectors of the intransitive group of rotations, but the three vectors η^i_α are not the vectors of a three-parameter group, since $\gamma^{\epsilon'}_{\alpha\beta} \neq 0$.

3. If ξ^i_α ($\alpha = 1, 2, 3$) are the vectors of a sub-group G_3, they are linear combinations, with constant coefficients, of the vectors η^i_θ, and, if G_3 is transitive, then the vectors η^i_α must occur independently. It can therefore be assumed without loss of generality that the parameters of G_3 are chosen so that the vectors are given by

$$\xi^i_\alpha = \eta^i_\alpha + t^\lambda_\alpha \eta^i_\lambda, \tag{5}$$

where the t's are constants to be determined.

The equations to be satisfied by ξ^i_α in order that they shall be the vectors of a group are

$$\xi^k_\alpha \xi^i_{\beta.k} - \xi^k_\beta \xi^i_{\alpha.k} = C^\epsilon_{\alpha\beta}\xi^i_\epsilon,$$

where the C's are some constants, the constants of structure for the group. Substituting from (5), (3), (4), these equations lead to linear relations between η^i_θ. It is, however, known that the vectors of a

† Loc. cit. 512. I have written $\eta^i_{\alpha'}$ for Robertson's $\xi^i_{\alpha'}$.

group are linearly independent (constant coefficients) whence, equating to zero the coefficients of η_ϵ^i and η_ν^i, we find

$$C_{\alpha\beta}^{\epsilon} = t_\epsilon^\lambda e_{\lambda\alpha\beta} - t_\lambda^\lambda e_{\epsilon\alpha\beta}, \tag{6}$$

$$t_\epsilon^\nu C_{\alpha\beta}^{\epsilon} = -t_\alpha^\lambda t_\beta^\mu e_{\nu\lambda\mu} - K e_{\nu\alpha\beta}. \tag{6'}$$

Substituting from (6) in (6'), multiplying through by $\frac{1}{2}e_{\alpha\beta\gamma}$ and summing for α and β, we finally have, after changing suffixes,

$$t_\epsilon^\alpha t_\epsilon^\beta + \bar{t}_\beta^\alpha - t_\epsilon^\epsilon t_\beta^\alpha + K\delta_\beta^\alpha = 0, \tag{7}$$

where $\bar{t}_\beta^\alpha$ is the cofactor of t_β^α in the determinant $|t_\mu^\lambda|$. These, then, are the equations to be satisfied by t_α^λ.

4. The general solution of (7) can be found by noting that the form of these equations is unaltered by an orthogonal transformation $t_\beta^\alpha = p_\lambda^\alpha \tau_\mu^\lambda p_\mu^\beta$, where the p's are constants satisfying

$$p_\alpha^\epsilon p_\beta^\epsilon = p_\epsilon^\alpha p_\epsilon^\beta = \delta_\beta^\alpha, \qquad |p_\beta^\alpha| = 1.$$

The equations for the τ's become

$$\tau_\epsilon^\alpha \tau_\epsilon^\beta + \bar{\tau}_\beta^\alpha - \tau_\epsilon^\epsilon \tau_\beta^\alpha + K\delta_\beta^\alpha = 0, \tag{8}$$

and we can choose the p's so that the symmetric part of (t_β^α) transforms into the simple form with only diagonal terms. The skew-symmetric part remains skew-symmetric under the transformation, and we may therefore assume for (τ_μ^λ) the form

$$\tau_1^1 = \tau_1, \qquad \tau_2^2 = \tau_2, \qquad \tau_3^3 = \tau_3,$$

$$\tau_3^2 = -\tau_2^3 = \theta_1, \qquad \tau_1^3 = -\tau_3^1 = \theta_2, \qquad \tau_2^1 = -\tau_1^2 = \theta_3.$$

Substituting in (8) we find that

$$\tau_i \theta_j = 0 \quad (i \neq j),$$

$$\tau_2\tau_3 = \tau_3\tau_1 = \tau_1\tau_2 = \theta_1^2 + \theta_2^2 + \theta_3^2 + K,$$

and there are thus three cases to be considered:

(I) $\qquad \theta_1 = \theta_2 = \theta_3 = 0, \qquad \tau_1 = \tau_2 = \tau_3 = \pm K^{\frac{1}{2}};$

(II) $\qquad \theta_2 = \theta_3 = \tau_2 = \tau_3 = 0, \qquad \theta_1 = \pm(-K)^{\frac{1}{2}}, \qquad \tau_2 = \tau,$

where τ is arbitrary, and two other similar solutions:

(III) $\qquad \tau_1 = \tau_2 = \tau_3 = 0, \qquad \theta_1^2 + \theta_2^2 + \theta_3^2 = -K.$

In case (III) a further orthogonal transformation can be found to reduce this to case (II) with $\tau = 0$. Thus (I) and (II) cover all possibilities. We see at once that, for real sub-groups, (I) applies only when $K > 0$, and (II) only when $K < 0$, neglecting the case $K = 0$.

Returning now to t^α_β from τ^λ_μ by a general orthogonal transformation, we have

$$\text{(I)} \qquad t^\alpha_\beta = \pm K^{\frac{1}{2}}\delta^\alpha_\beta, \tag{9}$$

$$\text{(II)} \qquad t^\alpha_\beta = -(-K)^{\frac{1}{2}}a^\epsilon e_{\epsilon\alpha\beta} + \tau a^\alpha a_\beta, \tag{10}$$

where τ, a^α are arbitrary subject to $a^\alpha a^\alpha = 1$. A change of sign of the square root in (II) is equivalent to reversing a^α, so that we can fix the sign as indicated without loss of generality.

When $K > 0$ (I), there are two possible groups, with vectors

$$\xi^i_\alpha = \eta^i_\alpha \pm K^{\frac{1}{2}}\eta^i_{\alpha'}, \tag{11}$$

and from (6) the constants of structure for each are

$$C^\epsilon_{\alpha\beta} = -2e_{\epsilon\alpha\beta}.$$

It can be verified that these groups are invariant; they are, in fact, the sub-groups given by Robertson, and we have shown that there are no other real transitive three-parameter sub-groups.

When $K < 0$ (II), there are many sub-groups, each being characterized by a unit vector a^α and a number τ. The absence of spherical symmetry due to the vector a^α shows that these sub-groups are not invariant, the transformations of G_6 not belonging to each sub-group being those of the rotation group. The sub-groups found here are, however, of some interest in relativity and perhaps also in geometry, and a study of them will occupy the greater part of the remainder of this paper.

Substituting from (10) in (6), we find that the constants of structure for G_3 are

$$C^\epsilon_{\alpha\beta} = (-K)^{\frac{1}{2}}(a^\alpha\delta^\epsilon_\beta - a^\beta\delta^\epsilon_\alpha) + \tau(a^\epsilon a^\lambda e_{\lambda\alpha\beta} - e_{\epsilon\alpha\beta}). \tag{12}$$

It may be of interest to mention that when $\tau = 0$, any two linear combinations of the three vectors ξ^λ_α are vectors of a two-parameter group, which is, of course, a sub-group of G_3. This fact can easily be verified, for an equation similar to (3) is satisfied by such vectors as a consequence of (12).

5. The connexion coefficients for the group G_3 are defined by

$$\Lambda^i_{jk} = -\xi^\alpha_k \xi^i_{\alpha.j} \quad (\xi^i_{\alpha.k} + \xi^i_\alpha \Lambda^i_{kj} = 0), \tag{13}$$

where ξ^α_k is the normalized cofactor of ξ^k_α in $|\xi^i_\alpha|$. It is well known that the equations

$$\zeta^i{}_{.k} + \zeta^j \Lambda^i_{jk} = 0 \tag{14}$$

for the vector ζ^i are completely integrable, giving three independent

vector solutions. A vector field $\zeta^i(x)$ is fixed by an arbitrary vector $(\zeta^i)_0$ at one point, say the origin O $(x^i = 0)$, so that a triad $\zeta^i_a(x)$ is fixed by an arbitrary triad $(\zeta^i_a)_0$ at O.

The vectors ζ^i_a are well known as the fundamental vectors of a distant parallelism in S_3, and they form a *base* at each point of the space. The base at a point serves to fix the orientation of the coordinate system with that point as origin, e.g. coordinates may be defined so that the base vectors at a point give the parametric directions at that point, now the origin. Thus the complete solution of (14) gives the orientation at each point of the space when the orientation at one particular point is specified. The significance of the field of solutions is that the transformation relating the two coordinate systems determined by the bases at any two points belongs to the group G_6.

We shall call any solution of (14) a *base vector*, and talk of the vector field derived from a vector λ^i at O, meaning the field $\zeta^i(x)$ determined by $\zeta^i(0) = \lambda^i$.

The *paths* of the group G_3 are the curves given by the equations

$$\frac{d^2x^i}{d\theta^2} + \Lambda^i_{jk}\frac{dx^j}{d\theta}\frac{dx^k}{d\theta} = 0, \tag{15}$$

where θ is a parameter, and from (13) and (14) these equations admit two first integrals

$$\text{(i)}\quad \frac{dx^i}{d\theta} = p^\alpha \xi^i_\alpha, \qquad \text{(ii)}\quad \frac{dx^i}{d\theta} = q^a \zeta^i_a, \tag{16}$$

where p^α, q^a are arbitrary constants.

From (i) we have that each path is the path of a one-parameter sub-group; it is possible to choose bases on a curve in S_3 so that the resulting coordinate systems are related by transformations of a group G_1, a sub-group of G_3, only if the curve is one of the above paths. A member of G_1 corresponds to a value of the parameter θ, and, when two transformations are combined, the parameters are added.

From (ii) we have that the same curves are also the paths defined by the fields of base vectors. Each vector field gives a set of paths covering S_3, and we talk of the paths derived from a vector λ^i at O meaning those given by $dx^i/d\theta = \zeta^i(x)$, where $\zeta^i(0) = \lambda^i$.

It is known that, if g_{ij} is the fundamental tensor of S_3 and if ζ^i_a, ζ^i_b are any two base vectors, then $g_{ij}\zeta^i_a\zeta^j_b$ is constant over the field.

In particular, the magnitude of each base vector is constant, whence it follows from (ii) that the parameter θ is proportional to the arc distance along the path.

We shall now proceed to calculate the coefficients Λ^i_{jk} and the base vectors, and to determine also the paths in most cases.

6. (I) ($K > 0$). Considering the group where $\xi^i_\alpha = \eta^i_\alpha + K^{\frac{1}{2}}\eta^i_{\alpha'}$, it can be verified that the covariant vectors ξ^α_i are

$$\xi^\alpha_i = (1+\tfrac{1}{4}Kr^2)^{-2}\xi^i_\alpha,$$

and from (13) we find that

$$\Lambda^i_{jk} = \Gamma^i_{jk} + K^{\frac{1}{2}}(1+\tfrac{1}{4}Kr^2)^{-1}e_{ijk}, \tag{17}$$

where Γ^i_{jk} are the Christoffel symbols calculated from the metric (1). Substituting in (13), it is at once seen that the equations for ζ^i are similar to (13) for ξ^i_α but with the sign of $K^{\frac{1}{2}}$ changed. We can therefore write

$$\zeta^i_a = \eta^i_a - K^{\frac{1}{2}}\eta^i_{\alpha'}. \tag{18}$$

It is known that the base vectors are the vectors of another group, said to be *reciprocal* to the first group. In the present case, therefore, each of the two sub-groups of G_6 is the reciprocal of the other, as found by Robertson.

As remarked in § 5, the vectors ζ^i_a determine at each point of S_3 the orientation of the base fixed by the group under consideration. The orientation of the base at a point P can be conveniently compared with that at O by displacing the base at O by 'parallel transport' (Levi-Civita) from O to P. If Q is a point, of coordinates dx^i, near O and if λ^i is a base vector at O, then the parallel vector at Q is also, to the first order, λ^i, since $\Gamma^i_{jk} = 0$ at O. From (18) and (2) the base vector at Q is $\lambda^i + K^{\frac{1}{2}}e_{ijk}\lambda^j\,dx^k$, which can be obtained from λ^i by a rotation $K^{\frac{1}{2}}\,ds$ about OQ. Finally, integrating along a geodesic OP, we have that, if O, P are any two points, then the base at P is obtained from the base at O by first rotating the latter through the angle $K^{\frac{1}{2}}s$ about the geodesic OP and then displacing it by parallel transport along this geodesic, s being the geodesic distance OP.

When (17) is substituted in (15), these equations become

$$\frac{d^2x^i}{d\theta^2} + \Gamma^i_{jk}\frac{dx^j}{d\theta}\frac{dx^k}{d\theta} = 0.$$

Thus the paths are geodesics in S_3, the parameter being proportional to the arc as shown generally in § 5.

7. (II) ($K < 0$). In this case the constant unit vector a^α appears, and we could, if desired, refer to the particular coordinate system for which $a^1 = 1$, $a^2 = a^3 = 0$. It is, I believe, easier, however, to retain a^α and refer to an unrestricted coordinate system.

We shall write $k = 2(-K)^{-\frac{1}{2}}$, and

$$X^i = x^i + ka^i, \qquad R^2 = X^i X^i,$$
$$A^i = \frac{2}{R^2}(a^\lambda X^\lambda)X^i - a^i. \tag{19}$$

Then, from (5), (10), (2), the vectors of the group are

$$\xi^i_\alpha = k^{-2}(R^2\delta^i_\alpha - 2x^\alpha X^i) + \tau a^\alpha e_{i\lambda\mu} a^\lambda X^\mu,$$

and it can be shown that the covariant vectors are

$$\xi^\alpha_i = \frac{1}{R^2}\left\{k^2\delta^\alpha_i + \left(1-\frac{r^2}{k^2}\right)^{-2}(2x^i X^\alpha - k^2\tau A^i e_{\alpha\lambda\mu} a^\lambda X^\mu)\right\}.$$

Hence the connexion coefficients are

$$\Lambda^i_{jk} = \frac{2}{R^2}(\delta^\alpha_k X^i + \delta^i_k X^j) + \left(1-\frac{r^2}{k^2}\right)^{-1}\left(\frac{2}{k^2}\delta^i_j x^k + \tau e_{ij\lambda} A^\lambda A^k\right)$$
$$= \Gamma^i_{jk} + \left(1-\frac{r^2}{k^2}\right)^{-1}\left(\frac{2}{k}(\delta^i_k A^i - \delta^i_k A^j) + \tau e_{ij\lambda} A^\lambda A^k\right). \tag{20}$$

When dealing with base vectors it is convenient to refer to the flat space E_3 with coordinates x^i and metric $dx^i dx^i$. A contravariant vector λ^i in S_3 then corresponds to a Euclidean vector $\mathbf{l}$ with components $(1-r^2/k^2)^{-1}(\lambda^1, \lambda^2, \lambda^3)$. The magnitude and direction of λ^i in S_3 are given correctly by the magnitude $(\mathbf{l}^2)^{\frac{1}{2}}$ and direction of $\mathbf{l}$ in E_3, and the inner product $(g_{ij}\lambda^i\mu^j)$ of two vectors in S_3 is given by the scalar product in E_3. The base vector in E_3 corresponding to ζ^i is given by

$$\mathbf{z} = (1-r^2/k^2)^{-1}(\zeta^1, \zeta^2, \zeta^3). \tag{21}$$

We shall now write

$$\mathbf{a} = (a^1, a^2, a^3), \qquad \mathbf{r} = (x^1, x^2, x^3),$$
$$\mathbf{R} = (X^1, X^2, X^3), \qquad \mathbf{A} = (A^1, A^2, A^3),$$

whence, from (19),

$$\mathbf{R} = \mathbf{r} + k\mathbf{a}, \qquad \mathbf{A} = \frac{2}{R^2}(\mathbf{a}\cdot\mathbf{R})\mathbf{R} - \mathbf{a}. \tag{22}$$

Since $\mathbf{a}^2 = 1$, it follows that $\mathbf{A}^2 = 1$. From (14), (20), (21) it can

now be shown that the base vectors are given by the total differential equation

$$d\mathbf{z} = \frac{2}{R^2}\mathbf{z}\times(d\mathbf{R}\times\mathbf{R})+\tau\left(1-\frac{r^2}{k^2}\right)^{-1}(\mathbf{A}\cdot d\mathbf{R})\mathbf{A}\times\mathbf{z}. \tag{23}$$

We know this equation is completely integrable, and that the scalar product of any two solutions is constant.

8. In discussing equation (23) the first thing to note is that $\mathbf{z} = \mathbf{A}$ is a particular solution. This can easily be verified and appears again in § 9 since τ does not affect this solution. At O, $\mathbf{r} = 0$ and $\mathbf{A}$ becomes $\mathbf{a}$, so that the vector field $\mathbf{A}$ is derived from the base vector $\mathbf{a}$ at O.

The expression for $\mathbf{A}$ in terms of $\mathbf{R}$ and $\mathbf{a}$ shows that the direction of $\mathbf{A}$ is found by reflecting $\mathbf{a}$ in $\mathbf{R}$. Hence the vector field $\mathbf{A}$ in E_3 consists of the tangents to all the circles which pass through the point $-k\mathbf{a}$ and touch the line joining this point to O. These circles are therefore the paths determined by the base vector $\mathbf{a}$ at O.

The point $-k\mathbf{a}$ in E_3, i.e. the point $-ka^i$ in S_3, plays an important part in the geometrical study of the group G_3. It lies on the boundary $r^2 = k^2$ and may be regarded as the same geometric point wherever the origin is taken. We shall refer to it as the point I.

The particular path through O is the line IO in E_3, and this corresponds in S_3 to the geodesic $x^i \propto a^i$, through O. Since O is a typical point of S_3, it follows that all the paths derived from the base vector a^i at O are geodesics.

All the above circles in E_3 are orthogonal to the spheres through I with centres on IO. The paths derived from a^i at O are thus the orthogonal trajectories (in E_3 and S_3) of the family of surfaces $\mathbf{R}^2 = \lambda\mathbf{a}\cdot\mathbf{R}$, λ being a parameter.

9. We shall now find the general solution of (23) with $\tau = 0$, i.e. the equation

$$d\mathbf{z} = \frac{2}{R^2}\mathbf{z}\times(d\mathbf{R}\times\mathbf{R}). \tag{24}$$

We find

$$d\left(\frac{\mathbf{z}\cdot\mathbf{R}}{R^2}\right) = -\frac{\mathbf{z}\cdot d\mathbf{R}}{R^2},$$

whence, substituting in (24),

$$d\mathbf{z} = 2\frac{\mathbf{z}\cdot\mathbf{R}}{R^2}d\mathbf{R}+2d\left(\frac{\mathbf{z}\cdot\mathbf{R}}{R^2}\right)\mathbf{R} = 2d\left(\frac{\mathbf{z}\cdot\mathbf{R}}{R^2}\mathbf{R}\right).$$

Hence $$\mathbf{z} = \frac{2}{R^2}(\mathbf{z}\cdot\mathbf{R})\mathbf{R} - \boldsymbol{\gamma},$$

where $\boldsymbol{\gamma}$ is constant, and, by taking the scalar product with $\mathbf{R}$, $\mathbf{z}\cdot\mathbf{R} = \boldsymbol{\gamma}\cdot\mathbf{R}$, giving

$$\mathbf{z} = \frac{2}{R^2}(\boldsymbol{\gamma}\cdot\mathbf{R})\mathbf{R} - \boldsymbol{\gamma}, \tag{25}$$

where $\boldsymbol{\gamma}$ is an arbitrary constant vector. This, then, is the general solution. When $\boldsymbol{\gamma} = \mathbf{a}$, we have the previous solution $\mathbf{z} = \mathbf{A}$.

To find the vector field derived from a unit base vector $\mathbf{c}$ at O, we write $\mathbf{r} = 0$, $\mathbf{z} = \mathbf{c}$, and find

$$\boldsymbol{\gamma} = 2(\mathbf{c}\cdot\mathbf{a})\mathbf{a} - \mathbf{c}. \tag{26}$$

Thus, in E_3, $\boldsymbol{\gamma}$ is the reflection of $\mathbf{c}$ in $\mathbf{a}$, and $\mathbf{z}$ at the point $\mathbf{r}$ is the reflection of $\boldsymbol{\gamma}$ in $\mathbf{R}$. It follows that the vector field derived from $\mathbf{c}$ at O consists of the tangents to all the circles in E_3 which pass through I and whose tangents at I are in the direction $\boldsymbol{\gamma}$, the reflection of $\mathbf{c}$ in $\mathbf{a}$. These circles are therefore the paths derived from the base vector $\mathbf{c}$ at O. They are orthogonal to the spheres through I and with centres on the line through I in the direction $\boldsymbol{\gamma}$, so that, in E_3 and S_3, these paths are the orthogonal trajectories of the family of surfaces $\mathbf{R}^2 = \lambda\boldsymbol{\gamma}\cdot\mathbf{R}$.

We can see at once that, except in the case $\mathbf{c} = \mathbf{a}$ considered in § 8, the paths are not geodesics in S_3. All geodesics through O in S_3 are straight lines in E_3, and the path from I to O is a straight line only when $\mathbf{c} = \mathbf{a}$. The conclusion follows since O is a typical point. The base vectors of a field are, however, parallel (Levi-Civita) in S_3 along the particular paths derived from $\mathbf{a}$, for from (14) the condition for this is $(\Lambda^i_{jk} - \Gamma^i_{jk})A^k = 0$, which is satisfied when $\tau = 0$, as can be seen from the alternative form in (20).

The space S_3 corresponds in E_3 to the inside of the sphere $r^2 = k^2$, and each path is, in E_3, a circle which cuts the boundary at I and one other point, the path being that part of the circle inside the boundary. Although each path is associated with a group G_1, we can see that the boundary is never reached by finite combinations in the group, for the parameter of the group is additive. This parameter is proportional to the arcual distance along the path, and the arc tends to infinity as the boundary is approached.

The orientation of the base at P can be compared with that at O, as in § 6, by comparing at P the base there with the base at O

displaced by parallel transport along the geodesic from O to P. By the method used in § 6 it can be shown that, if the base at O is rotated through an angle θ about an axis in the direction $\mathbf{a}\times\mathbf{d}$, where $\mathbf{d}$ is the direction of the geodesic OP, α the angle between $\mathbf{a}$ and $\mathbf{d}$, s the geodesic distance OP, and†

$$\cot\tfrac{1}{2}\theta = \cot\alpha + \operatorname{cosec}\alpha\coth\frac{s}{k},$$

then the base so obtained coincides with that at P when displaced by parallel transport along the geodesic from O to P.

We observe that there is no rotation when passing from O along the geodesic in the direction $\mathbf{a}$, and that the angle of rotation is simply $2\tan^{-1}(\tanh s/k)$ along a geodesic orthogonal to $\mathbf{a}$ at O. The orientation at P can, if desired, be studied by travelling from O to P first along a geodesic orthogonal to $\mathbf{a}$ at O, and then along one of the paths derived from $\mathbf{a}$ at O.

The bases throughout S_3 obtained by displacing the base at O by parallel transport along all the geodesics through O are familiar in that they are given by the standard Lorentz transformations, applied to another coordinate system suitable for space of negative curvature.‡ It is usual to consider only transformations in which the origin moves along a geodesic in S_3 and the 'axes' are not 'rotated', and these certainly form a group G_1 which is a sub-group of G_6. If, however, this is done for all geodesics through a fixed point O, the three-parameter set of transformations so obtained do not form a group as one might assume; some rotation is necessary after each transport if the new bases are required to belong to a group.

10. Returning now to (23) with any value of τ, we shall consider at each point the triad $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, where $\mathbf{A}$ is the vector already defined and $\mathbf{B}$, $\mathbf{C}$ are the solutions of (24) derived from vectors $\mathbf{b}$, $\mathbf{c}$ at O such that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form a right-handed orthogonal set of unit vectors. Then $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ also form a right-handed orthogonal set of unit vectors at each point. Since $\mathbf{A}$ is known to satisfy (23), it follows that any solution derived from a vector orthogonal to $\mathbf{a}$ at O will be orthogonal to $\mathbf{A}$, whence the solution of (23) derived from the vector $\mathbf{b}$ at O will be given by

$$\mathbf{z} = \mathbf{B}\cos\phi + \mathbf{C}\sin\phi,$$

† More simply, $\frac{1}{2}\theta$ is the angle between the vectors $\mathbf{a}$ and $\mathbf{R}$.

‡ A. G. Walker, loc. cit., § 8.

where ϕ is a function of x^i and $\phi = 0$ when $x^i = 0$. Substituting in (23) and remembering that

$$d\mathbf{B} = \frac{2}{R^2}\mathbf{B}\times(d\mathbf{R}\times\mathbf{R}),$$

and similarly for $\mathbf{C}$, and that $\mathbf{A}\times\mathbf{B} = \mathbf{C}$, etc., we find

$$d\phi = \tau\left(1-\frac{r^2}{k^2}\right)^{-1}\mathbf{A}\cdot d\mathbf{r}.$$

Finally, from (19), this equation can be integrated to give

$$\phi = \tfrac{1}{2}\tau k\log\left(\frac{R^2}{k^2-r^2}\right). \tag{27}$$

It can be verified that ϕ is constant over any one of the spheres orthogonal to the paths derived from $\mathbf{a}$ at O (see § 8). These spheres are geodesic parallels in S_3, and we find

$$\phi = \tau s, \tag{28}$$

where s is the geodesic distance from the sphere through O to the sphere through P, the point under consideration.

The above solution, together with the solution $\mathbf{A}$, are sufficient, for we know that the vector field derived from $\mathbf{c}$ at O is given by $\mathbf{A}\times(\mathbf{B}\cos\phi+\mathbf{C}\sin\phi)$, i.e. $\mathbf{C}\cos\phi-\mathbf{B}\sin\phi$. Any other vector field can be expressed in terms of the three particular solutions found above. The base at P when $\tau \neq 0$ is obtained by first finding the base when $\tau = 0$ and rotating it through the above angle ϕ about $\mathbf{A}$.

The paths when $\tau \neq 0$, except those already considered, are too complicated to be considered here; they will be mentioned again later.

11. Since the paths when $\tau = 0$ are circles through I, the geometrical figure will be simplified by inversion with respect to a sphere with centre at I. In order to leave the origin and all directions at the origin unaltered, we shall consider the conformal transformation given by inversion with respect to the sphere with centre I and radius k, followed by reflection in the plane through O perpendicular to OI. If, then, x^{*i} are the coordinates of the point corresponding to x^i, and $\mathbf{r}^*$ denotes (x^{*1}, x^{*2}, x^{*3}), the transformation from (x) to (x^*) is found to be

$$\mathbf{r}^* = \frac{k^2}{R^2}\left(\mathbf{r}+\frac{r^2}{k}\mathbf{a}\right) \tag{29}$$

and the reverse transformation is

$$\mathbf{r} = \frac{k^2}{R^{*2}}\left(\mathbf{r}^* - \frac{r^{*2}}{k}\mathbf{a}\right), \qquad \mathbf{R}^* = \mathbf{r}^* - k\mathbf{a}. \tag{30}$$

The metric (1) of S_3 becomes under the transformation

$$ds^2 = \frac{(d\mathbf{r}^*)^2}{\left(1 - \frac{2}{k}\mathbf{a}\cdot\mathbf{r}^*\right)^2}. \tag{31}$$

This form, which still shows the space of constant curvature as conformal to flat space, is known† but little used. For most purposes it has the disadvantages that it applies only to space of negative curvature and that the coordinate system does not display spherical symmetry about the origin, there being a preferential direction. The latter, as we have seen, is required for the study of sub-groups when the curvature is negative, and the above coordinate system simplifies the problem considerably. We may add, however, that it is considered worth while in the general problem to refer to the symmetric coordinates at first, since it is then easier to see exactly how preferential directions arise and enter into the results.

In dealing with the new coordinate system, it is convenient to refer to the flat space E_3^*, of metric $(d\mathbf{r}^*)^2$, which is thus conformal to S_3 and E_3. The boundary $r^2 = k^2$ transforms to become the surface

$$\mathbf{a}\cdot\mathbf{r}^* = \tfrac{1}{2}k, \tag{32}$$

and the boundary in E_3^* consists of this plane and the plane at infinity. The space S_3, corresponding to the inside of the sphere $r^2 = k^2$ in E_3, now corresponds to that part of E_3^* which is on the same side of the above plane as the origin, i.e. where $\mathbf{a}\cdot\mathbf{r}^* < \frac{1}{2}k$.

12. In § 9 it was shown that, when $\tau = 0$, the paths derived from the base vector $\mathbf{c}$ at O are, in E_3, circles through I, the direction at I being the reflection of $\mathbf{c}$ in $\mathbf{a}$. After inverting and reflecting as in § 11, these circles become straight lines in the direction $\mathbf{c}$, whence in E_3^* the paths derived from $\mathbf{c}$ at O are straight lines in the direction $\mathbf{c}$. In particular, whatever the value of τ, the paths derived from $\mathbf{a}$ are the lines in the direction $\mathbf{a}$, i.e. perpendicular to the bounding plane.

It follows from the above property that the vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ at

† Eisenhart, *Riemannian Geometry*, § 27. The present metric is given by $e_i = 1$, $a = 0$, $\sum c_i = 1$, $b_i = -a^i k$.

a point P in E_3, occurring in § 10, correspond, in E_3^*, to the constant vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ from which they were derived. Hence, when $\tau \neq 0$, the vector field derived from $\mathbf{b}$ at O is $\mathbf{b}\cos\phi + \mathbf{c}\sin\phi$, where ϕ, given by (27), can now be written

$$\phi = -\tfrac{1}{2}k\tau \log\left(1-\frac{2}{k}\mathbf{a}\cdot\mathbf{r}^*\right). \tag{33}$$

This angle is still given by (28), where s is now the geodesic distance (in S_3) between the planes through O and P parallel to the bounding plane in E_3^*.

The vectors, connexion coefficients, etc., for the new coordinate system can be obtained by direct transformation from those for the system x^i. The group vectors become ξ_α^{*i} where $\xi_\alpha^{*i} = \xi_\alpha^j \partial x^{*i}/\partial x^j$, and we find

$$\xi_\alpha^{*i} = \delta_\alpha^i - \frac{2}{k}a^\alpha x^{*i} + \tau a^\alpha e_{ijk} a^j x^{*k}. \tag{34}$$

From (13), or by transformation from (20), the connexion coefficients are

$$\Lambda_{jk}^{*i} = \left(1-\frac{2}{k}\mathbf{a}\cdot\mathbf{r}^*\right)^{-1}\left(\frac{2}{k}\delta_j^i + \tau e_{ij\lambda}a^\lambda\right)a^k. \tag{35}$$

The base vectors ζ^{*i} are given by equations of the form (14), and writing

$$\mathbf{z}^* = \left(1-\frac{2}{k}\mathbf{a}\cdot\mathbf{r}^*\right)^{-1}(\zeta^{*1}, \zeta^{*2}, \zeta^{*3}), \tag{36}$$

so that $\mathbf{z}^*$ is the corresponding Euclidean vector in E_3^*, we find that the equation for $\mathbf{z}^*$ is

$$d\mathbf{z}^* = \frac{\tau\mathbf{a}\cdot d\mathbf{r}^*}{1-\frac{2}{k}\mathbf{a}\cdot\mathbf{r}^*}.\,\mathbf{a}\times\mathbf{z}^*. \tag{37}$$

When $\tau = 0$, we have $z^* =$ constant, as was to be expected, since we have already seen that the paths are parallel straight lines. The general result for $\tau \neq 0$, given above, can also be obtained from (37).

When $\tau \neq 0$, the paths are too complicated to be considered here in detail. We note, however, that the angle with $\mathbf{a}$ is constant, showing that each path is a helix in E_3^*. It can be verified that each path is in fact a helix on a cylinder with an equiangular spiral cross-section.

13. The sub-groups of G_6 have so far been considered from a geometrical point of view, but we shall in conclusion regard the subject in the light of relativistic cosmology. In such theories, given by

general relativity or by Milne's kinematical relativity, a space S_3 of constant curvature plays an important part as the spatial part of a space-time continuum. An observer is assumed situated at each point of S_3, and these observers choose exactly similar coordinate systems, the transformation from one observer to another then being a motion of S_3 into itself, i.e. a member of G_6.

Each observer may be regarded as having orientation, determined by a base at the observer's position, and, if these bases are given by a sub-group G_3 of the kind considered in the present paper, it follows that the whole set appears the same from whatever point it is viewed, i.e. there are no preferential observers. This possible arrangement has not so far been considered by other writers, for it has always been assumed that there is complete spherical symmetry about each point, so that the question of orientation has not arisen. It may, however, be desirable to consider other possibilities, where spherical symmetry is discarded without implying the existence of preferential observers. In such cases the groups discussed in the present paper will be required.

The sharp distinction between spaces of positive and negative curvature where sub-groups are concerned is emphasized in the present paper and is, I believe, noteworthy. Although often discussed, it has not so far been decided which sign of curvature in a relativistic model leads to a closer comparison with reality, and it is conceivable that the properties of sub-groups will assist in indicating that one sign is to be preferred to the other. The conclusions of this paper are that when the curvature is positive, there are two sub-groups, each avoiding preferential directions, but so similar that there could never be any theoretical reason for choosing one against the other. When the curvature is negative, there is a preferential direction at each point, but this does not imply any preferential points in S_3.

SOME INTEGRALS INVOLVING LEGENDRE AND BESSEL FUNCTIONS

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1. Introductory

In § 2 of this paper three integrals involving associated Legendre functions are evaluated in terms of hypergeometric functions. In each case an expression for an integral of this type in terms of gamma functions can be deduced by applying Gauss's theorem. In § 3 relations between these integrals and known integrals involving Bessel functions are elucidated. In § 4 some integrals involving associated Legendre functions and in § 5 corresponding integrals involving Bessel functions are evaluated in terms of E-functions.

2. Integrals involving associated Legendre functions

Take first the integral

$$\int_0^\infty (\sinh u)^{l+m-1}(x^2\cosh^2 u-1)^{-\frac{1}{2}m}Q_n^m(x\cosh u)\,du$$
$$=\frac{\Gamma(\frac{1}{2}n+\frac{1}{2}m+\frac{1}{2})\Gamma(\frac{1}{2}l+\frac{1}{2}m)\Gamma(\frac{1}{2}n-\frac{1}{2}l+1)}{2^{2-m}\Gamma(n+\frac{3}{2})x^{n+m+1}}\times$$
$$\times F\left[\begin{matrix}\frac{1}{2}n+\frac{1}{2}m+\frac{1}{2},\ \frac{1}{2}n-\frac{1}{2}l+1;\\ n+\frac{3}{2}\end{matrix}\ \frac{1}{x^2}\right], \quad (1)$$

where $|x|>1$, $\mathrm{R}(l+m)>0$, $\mathrm{R}(n-l+2)>0$. It can be established by expanding the Q-function in the integrand in descending powers of x and integrating term by term. The formula

$$\int_0^\infty \frac{(\sinh u)^{n-1}}{(\cosh u)^{m+n-1}}\,du=\tfrac{1}{2}\mathrm{B}(\tfrac{1}{2}m,\tfrac{1}{2}n), \quad (2)$$

where $\mathrm{R}(m)>0$, $\mathrm{R}(n)>0$ is required.

If, moreover, $\mathrm{R}(l-m)>0$, on making $x\to 1$ and applying Gauss's theorem it is found that

$$\int_0^\infty (\sinh u)^{l-1}Q_n^m(\cosh u)\,du$$
$$=\frac{\Gamma(\frac{1}{2}n+\frac{1}{2}m+\frac{1}{2})\Gamma(\frac{1}{2}n-\frac{1}{2}l+1)\Gamma(\frac{1}{2}l+\frac{1}{2}m)\Gamma(\frac{1}{2}l-\frac{1}{2}m)}{2^{2-m}\Gamma(\frac{1}{2}n-\frac{1}{2}m+1)\Gamma(\frac{1}{2}n+\frac{1}{2}l+\frac{1}{2})}, \quad (3)$$

where $\mathrm{R}(l\pm m)>0$, $\mathrm{R}(n-l+2)>0$.

The next integral is

$$\int_0^\infty (\sinh u)^{l+m-1}(x^2\cosh^2 u-1)^{-\frac{1}{2}m}P_n^{-m}(x\cosh u)\,du$$

$$=\frac{2^{-m-1}\Gamma(\frac{1}{2}l+\frac{1}{2}m)\Gamma(\frac{1}{2}n-\frac{1}{2}l+1)\Gamma(\frac{1}{2}-\frac{1}{2}l-\frac{1}{2}n)}{\Gamma(\frac{1}{2}m+\frac{1}{2}n+1)\Gamma(\frac{1}{2}m-\frac{1}{2}n+\frac{1}{2})\Gamma(\frac{1}{2}m-\frac{1}{2}l+1)x^{m+n+1}}\times$$

$$\times F(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{1}{2},\tfrac{1}{2}n-\tfrac{1}{2}l+1;\tfrac{1}{2}m-\tfrac{1}{2}l+1;1-x^{-2}),\quad(4)$$

where $|1-1/x^2|<1$, $\mathrm{R}(l+m)>0$, $\mathrm{R}(n-l+2)>0$, $\mathrm{R}(1-l-n)>0$. Formula (4) can be deduced from formula (1) by means of the relation

$$P_n^{-m}(z)=-\frac{\sec n\pi}{\Gamma(n+m+1)\Gamma(m-n)}\{Q_n^m(z)-Q_{-n-1}^m(z)\}.\quad(5)$$

On making $x\to 1$ in (4), we obtain the formula

$$\int_0^\infty (\sinh u)^{l-1}P_n^{-m}(\cosh u)\,du$$

$$=\frac{\Gamma(\frac{1}{2}l+\frac{1}{2}m)\Gamma(\frac{1}{2}n-\frac{1}{2}l+1)\Gamma(\frac{1}{2}-\frac{1}{2}l-\frac{1}{2}n)}{2^{m+1}\Gamma(\frac{1}{2}m+\frac{1}{2}n+1)\Gamma(\frac{1}{2}m-\frac{1}{2}n+\frac{1}{2})\Gamma(\frac{1}{2}m-\frac{1}{2}l+1)},\quad(6)$$

where $\mathrm{R}(l+m)>0$, $\mathrm{R}(n-l+2)>0$, $\mathrm{R}(1-l-n)>0$.

The integral

$$\int_0^\pi (\sin\theta)^{l+m-1}(1-x^2\cos^2\theta)^{-\frac{1}{2}m}T_n^{-m}(x\cos\theta)\,d\theta$$

$$=\frac{\pi 2^{-m}\Gamma(\frac{1}{2}l+\frac{1}{2}m)}{\Gamma(\frac{1}{2}m-\frac{1}{2}n+\frac{1}{2})\Gamma(\frac{1}{2}m+\frac{1}{2}n+1)\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2})}\times$$

$$\times F\left[\begin{matrix}\frac{1}{2}m-\frac{1}{2}n,\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2};\\ \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}\end{matrix}\,x^2\right],\quad(7)$$

where $|x|<1$, $\mathrm{R}(l+m)>0$ can be established by employing the formula

$$T_n^{-m}(z)=2^{-m}(1-z^2)^{\frac{1}{2}m}\times$$

$$\times\left\{\begin{aligned}&\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}m-\frac{1}{2}n+\frac{1}{2})\Gamma(\frac{1}{2}m+\frac{1}{2}n+1)}F(\tfrac{1}{2}m-\tfrac{1}{2}n,\tfrac{1}{2}m+\tfrac{1}{2}n+\tfrac{1}{2};\tfrac{1}{2};z^2)+\\&+\frac{\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}m-\frac{1}{2}n)\Gamma(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2})}\times\\&\qquad\times zF(\tfrac{1}{2}m-\tfrac{1}{2}n+\tfrac{1}{2},\tfrac{1}{2}m+\tfrac{1}{2}n+1;\tfrac{3}{2};z^2)\end{aligned}\right\},\quad(8)$$

where $|z|<1$.

The formula to which (7) reduces when $x \to 1$ was given in a previous paper.*

3. Connexions with Bessel-function integrals

On substituting from the formula†

$$\sqrt{(\tfrac{1}{2}\pi)}\Gamma(m+n+1)\Gamma(m-n)(z^2-1)^{-\frac{1}{4}m}P_n^{-m}(z) = \int_0^\infty e^{-\lambda z}K_{n+\frac{1}{2}}(\lambda)\lambda^{m-\frac{1}{2}}\,d\lambda, \tag{9}$$

where $\mathrm{R}(z) > -1$, $\mathrm{R}(m+n+1) > 0$, $\mathrm{R}(m-n) > 0$, the left-hand side of (4), multiplied by $\sqrt{(\frac{1}{2}\pi)}\Gamma(m+n+1)\Gamma(m-n)$, becomes

$$\int_0^\infty (\sinh u)^{l+m-1}\,du \int_0^\infty e^{-\lambda x\cosh u}K_{n+\frac{1}{2}}(\lambda)\lambda^{m-\frac{1}{2}}\,d\lambda$$

$$= \int_0^\infty K_{n+\frac{1}{2}}(\lambda)\lambda^{m-\frac{1}{2}}\,d\lambda \int_0^\infty e^{-\lambda x\cosh u}(\sinh u)^{l+m-1}\,du$$

$$= \frac{\Gamma(\frac{1}{2}l+\frac{1}{2}m)}{\Gamma(\frac{1}{2})}\left(\frac{2}{x}\right)^{\frac{1}{2}l+\frac{1}{2}m-\frac{1}{2}} \int_0^\infty K_{n+\frac{1}{2}}(\lambda)K_{\frac{1}{2}l+\frac{1}{2}m-\frac{1}{2}}(x\lambda)\lambda^{\frac{1}{2}m-\frac{1}{2}l}\,d\lambda,$$

where $\mathrm{R}(x) > -1$, $\mathrm{R}(m+n) > -1$, $\mathrm{R}(m-n) > 0$, $\mathrm{R}(n-l+1) > -1$, $\mathrm{R}(l+n) < 1$.

Here replace l, m, n by $m-l+\frac{3}{2}$, $l+m-\frac{1}{2}$, $n-\frac{1}{2}$ respectively. Then

$$\int_0^\infty K_m(x\lambda)K_n(\lambda)\lambda^{l-1}\,d\lambda$$

$$= \frac{\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}n)\Gamma(\frac{1}{2}l+\frac{1}{2}m-\frac{1}{2}n)\Gamma(\frac{1}{2}l-\frac{1}{2}m+\frac{1}{2}n)\Gamma(\frac{1}{2}l-\frac{1}{2}m-\frac{1}{2}n)}{2^{3-l}x^{l+n}\Gamma(l)} \times$$

$$\times F(\tfrac{1}{2}l+\tfrac{1}{2}m+\tfrac{1}{2}n, \tfrac{1}{2}l-\tfrac{1}{2}m+\tfrac{1}{2}n; l; 1-x^{-2}), \tag{10}$$

where $\mathrm{R}(x) > -1$, $|1-1/x^2| < 1$, $\mathrm{R}(l \pm m \pm n) > 0$.

This integral, as well as (12) below, has been discussed by Dixon and Ferrar,‡ who give references to earlier work. Formula (10) is due to Titchmarsh.§

Similarly, by substituting from the formula

$$\sqrt{\left(\frac{2}{\pi}\right)}(z^2-1)^{-\frac{1}{4}m}Q_n^m(z) = \int_0^\infty e^{-\lambda z}I_{n+\frac{1}{2}}(\lambda)\lambda^{m-\frac{1}{2}}\,d\lambda, \tag{11}$$

* T. M. MacRobert, *Phil. Mag.* 27 (1939), 703.

† Ibid. 704.

‡ A. L. Dixon and W. L. Ferrar, *Quart. J. of Math.* (Oxford), 1 (1930), 129.

§ E. C. Titchmarsh, *J. of London Math. Soc.* 2 (1926), 98.

where $\mathrm{R}(z) > 1$, $\mathrm{R}(m+n) > -1$ in the left-hand side of (1), we obtain the formula

$$\int_0^\infty K_m(\lambda x) I_n(\lambda) \lambda^{l-1}\, d\lambda = 2^{l-2} \frac{\Gamma(\frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}n)\Gamma(\frac{1}{2}l-\frac{1}{2}m+\frac{1}{2}n)}{\Gamma(n+1)x^{l+n}} \times \times F\left(\begin{matrix} \frac{1}{2}l+\frac{1}{2}m+\frac{1}{2}n, \frac{1}{2}l-\frac{1}{2}m+\frac{1}{2}n; \\ n+1 \end{matrix} x^{-2}\right), \quad (12)$$

where $\mathrm{R}(x) > 1$, $\mathrm{R}(l \pm m+n) > 0$.

4. Evaluation of some integrals involving Legendre functions in terms of E-functions

The results given in this and the following section are based on the two following formulae:

$$\int_0^\infty e^{-x\lambda} \lambda^{\gamma-1} F(\alpha, \beta; \delta; -\lambda)\, d\lambda = \frac{\Gamma(\delta) x^{-\gamma}}{\Gamma(\alpha)\Gamma(\beta)} E(\alpha, \beta, \gamma : \delta : x), \quad (13)$$

where $\mathrm{R}(x) > 0$, $\mathrm{R}(\gamma) > 0$,* and

$$\int_0^\infty e^{-x\lambda} \lambda^{\gamma-1} (1+\lambda)^{\alpha+\beta-\delta} F(\alpha, \beta; \delta; -\lambda)\, d\lambda = \frac{\Gamma(\delta) x^{-\gamma}}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)} E(\delta-\alpha, \delta-\beta, \gamma : \delta : x), \quad (14)$$

where $\mathrm{R}(x) > 0$, $\mathrm{R}(\gamma) > 0$. Formula (14) can be easily deduced from formula (13).

The integral

$$\int_1^\infty e^{-x\lambda} (\lambda-1)^{l-1} (\lambda^2-1)^{-\frac{1}{2}m} P_n^{-m}(\lambda)\, d\lambda = \frac{2^{-m} x^{-l}}{\Gamma(m+n+1)\Gamma(m-n)} e^{-x} E\left(\begin{matrix} m+n+1, m-n, l : \\ m+1 \end{matrix} 2x\right), \quad (15)$$

where $\mathrm{R}(x) > 0$, $\mathrm{R}(l) > 0$, can be derived by putting $\lambda = 1+2\mu$ and applying formula (13) or (14). The formula

$$\int_1^\infty e^{-x\lambda} (\lambda-1)^{l-1} (\lambda^2-1)^{\frac{1}{2}m} P_n^{-m}(\lambda)\, d\lambda = -\frac{\sin n\pi\, e^{-x}}{\pi x^{l+m}} E\left(\begin{matrix} -n, n+1, l+m : \\ m+1 \end{matrix} 2x\right), \quad (16)$$

where $\mathrm{R}(x) > 0$, $\mathrm{R}(l+m) > 0$, is derived in a similar manner.

* T. M. MacRobert, *Proc. Roy. Soc. Edinburgh*, 58 (1937), 10.

Again, from the relation

$$Q_n^m(\lambda) = \frac{\pi}{2\sin m\pi}\left\{P_n^m(\lambda) - \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}P_n^{-m}(\lambda)\right\}, \tag{17}$$

and formulae (15) and (16), the two following formulae can be deduced:

$$\int_1^\infty e^{-x\lambda}(\lambda-1)^{l-1}(\lambda^2-1)^{\frac{1}{2}m}Q_n^m(\lambda)\,d\lambda$$

$$= \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}e^{-x}\left\{\begin{array}{l}\dfrac{\sin n\pi}{2x^{l+m}\sin m\pi}E\left(\begin{matrix}-n, n+1, l+m: \\ m+1\end{matrix} 2x\right) - \\ \\ -\dfrac{\sin(m+n)\pi}{2^{1-m}x^l\sin m\pi}E\left(\begin{matrix}n-m+1, -n-m, l: \\ 1-m\end{matrix} 2x\right)\end{array}\right\}, \tag{18}$$

where $\mathrm{R}(x) > 0$, $\mathrm{R}(l) > 0$, $\mathrm{R}(l+m) > 0$; and

$$\int_1^\infty e^{-x\lambda}(\lambda-1)^{l-1}(\lambda^2-1)^{-\frac{1}{2}m}Q_n^m(\lambda)\,d\lambda$$

$$= -\frac{e^{-x}\sin n\pi}{2x^{l-m}\sin m\pi}E\left(\begin{matrix}-n, n+1, l-m; \\ 1-m\end{matrix} 2x\right) -$$

$$-\frac{e^{-x}\sin(m-n)\pi}{2^{m+1}x^l\sin m\pi}E\left(\begin{matrix}m+n+1, m-n, l: \\ m+1\end{matrix} 2x\right), \tag{19}$$

where $\mathrm{R}(x) > 0$, $\mathrm{R}(l) > 0$, $\mathrm{R}(l-m) > 0$.

5. Corresponding integrals involving Bessel functions

On substituting from (9) in (15), changing the order of integration, and evaluating the inner integral, it is found that

$$\Gamma(l)\int_0^\infty e^{-\mu}K_{n+\frac{1}{2}}(\mu)\mu^{m-\frac{1}{2}}(x+\mu)^{-l}\,d\mu$$

$$= \sqrt{(\tfrac{1}{2}\pi)}2^{-m}x^{-l}E\left(\begin{matrix}m+n+1, m-n, l: \\ m+1\end{matrix} 2x\right), \tag{20}$$

where $\mathrm{R}(x) > 0$, $\mathrm{R}(m+n+1) > 0$, $\mathrm{R}(m-n) > 0$.

Similarly, from (19) and (11) it can be deduced that

$$\sqrt{(2\pi)}\sin m\pi\,\Gamma(l)\int_0^\infty e^{-\mu}I_{n+\frac{1}{2}}(\mu)\mu^{m-\frac{1}{2}}(x+\mu)^{-l}\,d\mu$$

$$= -x^{m-l}\sin n\pi\,E(-n, n+1, l-m: 1-m: 2x) -$$

$$-2^{-m}x^{-l}\sin(m-n)\pi\,E(m+n+1, m-n, l: m+1: 2x), \tag{21}$$

where $\mathrm{R}(x) > 0$, $\mathrm{R}(m+n) > -1$, $\mathrm{R}(l-m) > 0$.

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Finally, from formula (24) of the paper referred to at the beginning of the section, and the formula

$$\sqrt{(\tfrac{1}{2}\pi)}\Gamma(m+n+1)\Gamma(m-n)(1-\xi^2)^{-\frac{1}{4}m}T_n^{-m}(\xi) = \int_0^\infty e^{-\lambda\xi}K_{n+\frac{1}{2}}(\lambda)\lambda^{m-\frac{1}{2}}\,d\lambda, \tag{22}$$

where $\mathrm{R}(m+n+1) > 0$, $\mathrm{R}(m-n) > 0$, which is identical with formula (9) above, it can be deduced that

$$\frac{2^m\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(m+n)\Gamma(m-n)}\int_0^\infty K_n(\lambda)(x-\lambda)^{-m}I_m(x-\lambda)\lambda^{m-1}\,d\lambda$$

$$= e^{-\frac{1}{2}i\pi}x^{-m}K_n(xe^{-i\pi}) + e^{(m+\frac{1}{2})i\pi}x^{-m}\frac{\cos n\pi}{\cos m\pi}K_n(x) +$$

$$+ \frac{e^{-x}\sin(m-n)\pi\sin(m+n)\pi}{x\cos(m\pi)2^{1-m}\sqrt{\pi}}E\left(\begin{matrix}1-m+n, 1-m-n, 1: \\ \frac{3}{2}-m\end{matrix} 2x\right), \tag{23}$$

where $\mathrm{R}(m\pm n) > 0$, $-\frac{1}{2}\pi < \arg x < \frac{3}{2}\pi$.

LINEAR PARTIAL DIFFERENTIAL EQUATIONS (II)

By T. W. CHAUNDY (*Oxford*)

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1. The general equation in two variables with distinct characteristics

My purpose here is to discuss, in general terms, the form taken by the solution of Cauchy's problem for the general linear partial differential equation in two independent variables in which *all the characteristics are distinct*. This restriction excludes all equations of parabolic type, though it excludes more than these: for differential equations with repeated characteristics need not always be parabolic,* in the same way that plane curves with coincident asymptotic directions may possess either parallel linear asymptotes or parabolic asymptotes. However, the condition of distinct characteristics makes for a conveniently simple classification.

If the differential equation is of order m, then, when $m > 2$, the characteristics are 'excessive', i.e. they exceed the number of independent variables. In (**2**), (**3**) I have already considered two classes of equation with excessive characteristics. Of these (**3**), the more recent, is more immediately relevant here.

When $m = 2$, certain simplifications can be made in the analysis, and, moreover, the independent variables can be changed to throw the equation into Riemann's form

$$s+\beta p+\alpha q+\gamma z = 0, \tag{1}$$

where α, β, γ are functions of the independent variables x, y; of this a full discussion has been given by Goursat,† and I shall therefore always tacitly presuppose that $m > 2$.

After explaining certain notations in § 2, I lay down in § 3 certain conditions to be satisfied by the set of m Green's functions (or 'fundamental solutions') $U_r(X, Y; x, y)$ of the mth-order equation. In the next three sections I show that Cauchy's problem can be solved when such a set of functions is known.

In the two types of equation with 'excessive' characteristics that I have previously discussed, namely equations with constant

* Some of the equations discussed in (**1**), (**2**) exemplify this.

† E. Goursat (**4**), chap. 26 (ii), i.e. pp. 33 *et seq.*

coefficients (2) and certain equations of hypergeometric type (3), it will be found, as might be expected, that the Green's functions there given conform to the general conditions laid down here in § 3, though I have not thought it necessary to give the details of verification here. However, as particular examples, they should suffice to show that these conditions imposed on the Green's functions, though adequate to secure a solution of Cauchy's problem, are not too stringent to be satisfied by possible functions.

2. Some notation

For convenience of notation I begin with the equation *adjoint* to the equation proposed for solution. Denote this adjoint by

$$\Phi U = 0, \tag{2}$$

where

$$\Phi \equiv \sum_{s=0}^{m} \phi^{(s)}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \tag{3}$$

and $\phi^{(s)}$ denotes a homogeneous polynomial of order s with (presumably) variable coefficients. Taking the polynomial of highest order m, suppose that

$$\phi^{(m)}(h, k) \equiv \prod_{r=1}^{m} (\eta_r h - \xi_r k). \tag{4}$$

Then a typical characteristic is defined by the differential equation

$$\xi_r dx + \eta_r dy = 0. \tag{5}$$

If the integral of this differential equation is

$$f_r(x, y) = \text{constant}, \tag{6}$$

then

$$\xi_r \Big/ \frac{\partial f_r}{\partial x} = \eta_r \Big/ \frac{\partial f_r}{\partial y}$$

and we can sufficiently suppose that

$$\xi_r = \frac{\partial f_r}{\partial x}, \qquad \eta_r = \frac{\partial f_r}{\partial y}, \tag{7}$$

for this requires (at most) that the differential equation (2) be multiplied through by some suitable factor. I suppose then that the equation has been 'prepared' in this way and therefore that (7) holds. This gives, in consequence,

$$\frac{\partial \eta_r}{\partial x} = \frac{\partial \xi_r}{\partial y}, \tag{8}$$

which leads to some simplification of analysis at a later stage.

Let us call such a characteristic as (6) a characteristic C_r. Differentiation along a characteristic C_r is effected by the operator

$$\Delta_r \equiv \eta_r \frac{\partial}{\partial x} - \xi_r \frac{\partial}{\partial y}. \tag{9}$$

I write further

$$\psi_r(h,k) \equiv \prod_{s \neq r} (\eta_s h - \xi_s k) = \phi^{(m)}(h,k)/(\eta_r h - \xi_r k) \tag{10}$$

and

$$\chi_r(h,k) \equiv \Delta_r \psi_r(h,k), \tag{11}$$

where, of course, if h, k are replaced, on the left, by functions of x, y, they are so replaced on the right *after* the operation Δ_r.

3. Conditions to be satisfied by a set of Green's functions

I now postulate the conditions to be satisfied by a set of Green's functions (or 'fundamental solutions')

$$U_r(X,Y;x,y) \quad (r = 1,\ldots, m). \tag{12}$$

(A) *They are solutions in variables (x,y) of the adjoint equation*

$$\Phi U = 0.$$

(B) *They satisfy the null condition*

$$\sum_{r=1}^{m} U_r = 0.$$

(C) *The Green's function U_r and its derivatives of order not exceeding $m-3$ all vanish along the associated characteristic*

$$f_r(x,y) = f_r(X,Y). \tag{13}$$

Thus, differentiating a derivative of U_r of order $m-3$ along the characteristic (13), we have

$$\left(\eta_r \frac{\partial}{\partial x} - \xi_r \frac{\partial}{\partial y}\right) \frac{\partial^{m-3} U_r}{\partial x^i \partial y^{m-3-i}} = 0,$$

i.e.

$$\eta_r \frac{\partial^{m-2} U_r}{\partial x^{i+1} \partial y^{m-3-i}} = \xi_r \frac{\partial^{m-2} U_r}{\partial x^i \partial y^{m-2-i}} \quad (i = 0,\ldots, m-3).$$

Hence

$$\frac{\partial^{m-2} U_r}{\partial x^s \partial y^{m-2-s}} = \xi_r^s \eta_r^{m-2-s} \theta_r \quad (s = 0,\ldots, m-2) \tag{14}$$

for some $\theta_r(X,Y;x,y)$. Only values of θ_r along the characteristic (13) interest us and these are defined as follows:

(D) *Along the characteristic* (13) *we define θ_r by the differential equation*

$$\Delta_r\{\theta_r \psi_r(\xi_r, \eta_r)\} = \theta_r\{\chi_r(\xi_r, \eta_r) - \phi^{(m-1)}(\xi_r, \eta_r)\}. \tag{15}$$

Since Δ_r is the operator which differentiates along the characteristic, this equation is effectively an ordinary differential equation and it defines θ_r completely except in respect of a multiplicative constant. We fix this constant by the further condition:

(E) *The values of the m* θ_r *at* (X, Y), *the point of concurrence of characteristics* (13), *satisfy the condition*

$$\sum_{r=1}^{m} \theta_r \psi_r(\xi_r, \eta_r) = 1. \tag{16}$$

We can state this condition (E) somewhat differently. For, if we sum (14) in r, we get, in virtue of the null condition (B),

$$\sum_{r=1}^{m} \theta_r \xi_r^s \eta_r^{m-2-s} = 0 \quad (s = 0,..., m-2)$$

for the values of θ_r at (X, Y).* Solving these $m-1$ equations for the ratios of the θ_r we get at once

$$\theta_r \propto 1/\psi_r(\xi_r, \eta_r),$$

since, by (10), $$\psi_r(\xi_r, \eta_r) = \prod_{s \neq r} (\xi_r \eta_s - \xi_s \eta_r).$$

Hence from (E) we get

$$m\theta_r(X, Y; X, Y) = 1/\psi_r(\xi_r, \eta_r). \tag{17}$$

If along a characteristic C_r we write

$$dx/\eta_r = -dy/\xi_r = dt_r,$$

then, by integrating the differential equation (15) along the characteristic (13) and using (17), we can combine the conditions (D), (E) into the single condition:

(F) *Along the characteristic* (13) *we define*

$$\theta_r(X, Y; x, y) = \frac{1}{m\psi_r(\xi_r, \eta_r)} \exp\left\{ \int_{(X,Y)}^{(x,y)} \frac{\chi_r(\xi_r, \eta_r) - \phi^{(m-1)}(\xi_r, \eta_r)}{\psi_r(\xi_r, \eta_r)} \, dt_r \right\}, \tag{18}$$

the integral being taken along the characteristic.

This completes the statement of the properties defining the Green's functions.

4. The solution of Cauchy's problem

We now introduce the 'given' equation

$$\overline{\Phi} V = 0 \tag{19}$$

* It is, of course, only at the point of concurrence of the m characteristics (13) that we can use *all* the m equations (14) simultaneously.

adjoint to the equation $\Phi U = 0$, which is the equation that we have so far been considering. By the defining properties of adjoints we can write, for arbitrary U, V,

$$V\Phi U - U\overline{\Phi}V = \frac{\partial G}{\partial y} - \frac{\partial H}{\partial x}, \tag{20}$$

where G, H are of the general form

$$\sum \gamma \frac{\partial^{a+b} U}{\partial x^a \partial y^b} \frac{\partial^{e+f} V}{\partial x^e \partial y^f}.$$

It is not easy to find a concise expression for them, but, if we put λ, μ for the operators $\partial/\partial x$, $\partial/\partial y$ acting on V and its derivatives and $-\lambda'$, $-\mu'$ for the same operators when they act on U and its derivatives, then we can write

$$G \equiv \Phi_x UV, \qquad H \equiv \Phi_y UV, \tag{21}$$

where Φ_x, Φ_y are operators of the type

$$\sum \gamma \lambda^e \mu^f (-\lambda')^a (-\mu')^b.$$

These expressions are not unique, for we can evidently replace Φ_x, Φ_y by

$$\Phi_x + \frac{\partial}{\partial x}\Theta, \qquad \Phi_y + \frac{\partial}{\partial y}\Theta,$$

where Θ is any similar operator of order not exceeding $m-2$. I shall return to this point later.

Now, by a fundamental theorem in the integral calculus, we have from (20)

$$\iint (V\Phi U - U\overline{\Phi}V)\, dxdy = \int (G\, dx + H\, dy), \tag{22}$$

where the double integral is taken over an arbitrary area, and the single integral is taken round the boundary of the area (with the proper convention of sign between the two integrals). Taking U to be the Green's function U_r, so that $\Phi U_r = 0$, we now have

$$\iint (U_r \overline{\Phi} V)\, dxdy + \int \{(\Phi_x U_r V)\, dx + (\Phi_y U_r V)\, dy\} = 0. \tag{23}$$

For this integrand I prescribe the path of integration of the line-integral as follows. Let P be an arbitrary point to which we assign the coordinates (X, Y) already appearing in $U_r(X, Y; x, y)$ but still at our disposal. Through P draw the characteristic (13), i.e.

$$f_r(x, y) = f_r(X, Y),$$

associated with U_r and let it cut in A_r the base-curve of Cauchy's

problem. On this base-curve take an arbitrary origin O and connect P to O by an arbitrary curve. Then I assign as path of integration of the line-integral in (22) the composite curve $OA_r PO$, which we may denote by (r), the enclosed area, denoted by $[r]$, being accordingly the region of integration of the double integral.

Finally, we sum in r the m equations (23), so that we consider

$$\sum_{r=1}^{m} \iint_{[r]} (U_r \overline{\Phi} V)\, dxdy + \sum_{r=1}^{m} \int_{(r)} \{(\Phi_x U_r V)\, dx + (\Phi_y U_r V)\, dy\} = 0. \quad (24)$$

Now the arc OP is common to all the integrals and along it U_r appears as $\sum U_r$, which vanishes in virtue of the null condition (13). Thus the arbitrary arc OP disappears from the line-integral.

5. The integral along a characteristic

Again the integral along the characteristic $A_r P$ will reduce in view of the properties of U_r along this characteristic. Splitting up Φ into its dimensional sets $\sum \phi^{(s)}$, we split up Φ_x, Φ_y into the corresponding $\sum \phi_x^{(s)}$, $\sum \phi_y^{(s)}$ (say), where $\phi_x^{(s)}$, $\phi_y^{(s)}$ do not involve differential operations of order exceeding $s-1$. Since, by (C), U_r and its derivatives of order not exceeding $m-3$ vanish along the characteristic, the parts of the integral arising from $\phi^{(s)}$ of order not exceeding $m-2$ vanish, and we are therefore left only with those terms which arise from $\phi^{(m-1)}$ and $\phi^{(m)}$. For simplicity I now temporarily suppress the suffix r.

Consider firstly the terms arising from $\phi^{(m-1)}$, namely

$$\int \{(\phi_x^{(m-1)} UV)\, dx + (\phi_y^{(m-1)} UV)\, dy\}. \quad (25)$$

Here, by the above argument, we retain only those terms which are of the highest order $m-2$ in the derivatives of U and which consequently do not involve derivatives of V. Thus we reduce the integral (25) to a form

$$\int V\{[\phi_x^{[m-1]}(\partial_x, \partial_y)U]\, dx + [\phi_y^{[m-1]}(\partial_x, \partial_y)U]\, dy\}, \quad (26)$$

where I now write ∂_x, ∂_y for $\partial/\partial x, \partial/\partial y$, and $\phi_x^{[m-1]}$, $\phi_y^{[m-1]}$ for the parts of $\phi_x^{(m-1)}$, $\phi_y^{(m-1)}$ of highest order in the operators λ', μ' which operate on U alone. But along the characteristic

$$\frac{dx}{\eta} = -\frac{dy}{\xi} \equiv dt, \quad \text{say}, \quad (27)$$

and, by (14),

$$\phi_x^{[m-1]}(\partial_x, \partial_y)U = \theta\phi_x^{[m-1]}(\xi, \eta), \quad \text{etc.}$$

Thus the integral reduces to

$$\int V\theta\{\eta\phi_x^{[m-1]}(\xi, \eta) - \xi\phi_y^{[m-1]}(\xi, \eta)\}\, dt. \tag{28}$$

Now, picking out from the adjoint identity (20) the terms arising from $\phi^{(m-1)}$, we have the identity

$$V\phi^{(m-1)}(\partial_x, \partial_y)U - U\overline{\phi}^{(m-1)}(\partial_x, \partial_y)V = \frac{\partial}{\partial y}\phi_x^{(m-1)}UV - \frac{\partial}{\partial x}\phi_y^{(m-1)}UV.$$

Regarding U, V in this identity as arbitrary and independent, we can pick out the coefficient of V and then, in this coefficient, the U-derivatives of highest order $m-1$. Isolating the operators and replacing ∂_x, ∂_y by h, k we shall then have the algebraic relation

$$\phi^{(m-1)}(h, k) = k\phi_x^{[m-1]}(h, k) - h\phi_y^{[m-1]}(h, k),$$

so that

$$\eta\phi_x^{[m-1]}(\xi, \eta) - \xi\phi_y^{[m-1]}(\xi, \eta) = \phi^{(m-1)}(\xi, \eta).$$

Thus, at length, the integral (28) reduces to

$$\int V\theta\phi^{(m-1)}(\xi, \eta)\, dt. \tag{29}$$

Coming to $\phi^{(m)}$ I define $\phi_x^{(m)}$, $\phi_y^{(m)}$ more exactly. In virtue of (10), (11) we have, still omitting the suffix r,

$$\phi^{(m)}(h, k) = (\eta h - \xi k)\psi(h, k),$$

$$\begin{aligned}\phi^{(m)}(\partial_x, \partial_y) &= (\eta\partial_x - \xi\partial_y)\psi(\partial_x, \partial_y) - \chi(\partial_x, \partial_y) \\ &= \partial_x[\eta\psi(\partial_x, \partial_y)] - \partial_y[\xi\psi(\partial_x, \partial_y)] - \chi(\partial_x, \partial_y),\end{aligned}$$

since $\partial_x\eta = \partial_y\xi$. Thus

$$\begin{aligned}&V\phi^{(m)}(\partial_x, \partial_y)U \\ &= \partial_x[V\eta\psi(\partial_x, \partial_y)U] - \partial_y[V\xi\psi(\partial_x, \partial_y)U] - (\Delta V)\psi(\partial_x, \partial_y)U - V\chi(\partial_x, \partial_y)U.\end{aligned} \tag{30}$$

We secure the complete 'exact-derivative' form $\partial_x[\ldots] + \partial_y[\ldots]$ in (30) either by adding the adjoint to the expression on the left or by adding the adjoints to the last two terms on the right. Either addition must lead to the same result since adjoints are unique. Hence

$$\begin{aligned}&V\phi^{(m)}(\partial_x, \partial_y)U - U\overline{\phi}^{(m)}(\partial_x, \partial_y)V \\ &\quad = -\partial_y[V\xi\psi(\partial_x, \partial_y)U + \psi_x(U\Delta V) + \chi_x(UV)] + \\ &\qquad + \partial_x[V\eta\psi(\partial_x, \partial_y)U + \psi_y(U\Delta V) + \chi_y(UV)],\end{aligned} \tag{31}$$

where ψ_x, ψ_y, χ_x, χ_y denote differential operators analogous to $\phi_x^{(s)}$, $\phi_y^{(s)}$ above.

Now ψ, χ like $\phi^{(m-1)}$ are of order $m-1$ and by a similar argument they contribute to the integral along AP expressions similar to (29), namely

$$\int (\Delta V)\theta\psi(\xi, \eta)\, dt, \qquad \int V\theta\chi(\xi, \eta)\, dt.$$

The remaining terms in (31) give in the integral

$$\int [V\psi(\partial_x, \partial_y)U](\xi\, dx + \eta\, dy),$$

and this vanishes since $\xi\, dx + \eta\, dy = 0$ along the characteristic.

Thus, at length, the contribution of $\phi^{(m)}$ to the integral along AP is

$$-\int \theta[(\Delta V)\psi(\xi, \eta) + V\chi(\xi, \eta)]\, dt,$$

and so, if we add in (29), the whole integral along AP comes out as

$$\int_A^P \theta[V\{\phi^{(m-1)}(\xi, \eta) - \chi(\xi, \eta)\} - (\Delta V)\psi(\xi, \eta)]\, dt.$$

Now, by condition (D), θ satisfies the differential equation (15) and so this integral is equal to

$$-\int_A^P \{V\Delta[\theta\psi(\xi, \eta)] + (\Delta V)\theta\psi(\xi, \eta)\}\, dt = -[V\theta\psi(\xi, \eta)]_A^P,$$

since, by (9), (27), $\qquad \Delta = \eta\dfrac{\partial}{\partial x} - \xi\dfrac{\partial}{\partial y},$

and, along AP, $\qquad dx = \eta\, dt, \qquad dy = -\xi\, dt.$

6. The solution completed

By (E), or, more precisely, by (17), the expression evaluated at P reduces to $-V(X, Y)/m$. Hence, finally, those parts of the line-integrals in (24) which arise from integration along the characteristics $A_r P$ reduce, in virtue of the properties of the Green's functions postulated in § 3, to

$$-V(X, Y) + \sum_{r=1}^{m} [\theta_r \psi_r(\xi, \eta) V]^{A_r}.$$

We have already seen that the parts of the line-integrals in (24) arising from integration along the arbitrary arc OP vanish in the summation, and thus there remain of the line-integrals only those

parts which arise from integration along the base-curve itself. We have therefore reduced (24) to the form

$$V(X,Y)=\sum_{r=1}^{m}[\theta_r\psi_r(\xi_r,\eta_r)V]^{A_r}+\sum_{r=1}^{m}\int_O^{A_r}\{(\Phi_x U_r V)\,dx+(\Phi_y U_r V)\,dy\}+$$

$$+\sum_{r=1}^{m}\iint_{[r]}(U_r\overline{\Phi}V)\,dxdy. \quad (32)$$

Here $$\Phi_x=\sum_{s=1}^{m}\phi_x^{(s)},$$

and, from (31),

$$\phi_x^{(m)}U_r V=-V\xi_r\psi_r(\partial_x,\partial_y)U_r-\psi_{rx}(U_r\Delta_r V)-\chi_{rx}(U_r V),$$

and so for Φ_y, $\phi_y^{(m)}U_r V$. Thus, more explicitly, we write (32) as

$$V(X,Y)=\sum_{r=1}^{m}[V\theta_r\psi_r(\xi_r,\eta_r)]^{A_r}+\sum_{r=1}^{m}\int_O^{A_r}-\Big\{[V\psi_r(\partial_x,\partial_y)U_r](\xi_r\,dx+\eta_r\,dy)+$$

$$+\Big[\Big(\sum_{s=1}^{m-1}\phi_x^{(s)}-\chi_{rx}\Big)U_r V-\psi_{rx}(U_r\Delta_r V)\Big]\,dx+$$

$$+\Big[\Big(\sum_{s=1}^{m-1}\phi_y^{(s)}-\chi_{ry}\Big)U_r V-\psi_{ry}(U_r\Delta_r V)\Big]\,dy\Big\}+$$

$$+\sum_{r=1}^{m}\iint_{[r]}(U_r\overline{\Phi}V)\,dxdy. \quad (33)$$

If $V(x,y)$ is any solution of the given equation $\overline{\Phi}V=0$, the double integral disappears and (33) then gives the solution of Cauchy's problem, for the values of V and its derivatives appearing on the right are all taken at points of the given base-curve $OA_1\ldots A_m$.

More generally, if the given equation has an absolute term, i.e. if it is

$$\overline{\Phi}V=f(x,y),$$

then (33) gives the solution of Cauchy's problem in the form

$$V(X,Y)=\sum_{r=1}^{m}[V\theta_r\psi_r(\xi_r,\eta_r)]^{A_r}+\sum_{r=1}^{m}\int_O^{A_r}-\Big\{[V\psi_r(\partial_x,\partial_y)U_r](\xi_r\,dx+\eta_r\,dy)+$$

$$+\Big[\Big(\sum_{s=1}^{m-1}\phi_x^{(s)}-\chi_{rx}\Big)U_r V-\psi_{rx}(U_r\Delta_r V)\Big]\,dx+$$

$$+\Big[\Big(\sum_{s=1}^{m-1}\phi_y^{(s)}-\chi_{ry}\Big)U_r V-\psi_{ry}(U_r\Delta_r V)\Big]\,dy\Big\}+$$

$$+\sum_{r=1}^{m}\iint_{[r]}U_r f(x,y)\,dxdy, \quad (34)$$

I should perhaps emphasize again that the $\phi_x^{(s)}$, $\phi_y^{(s)}$ in (33), (34) are not unique; but, if they are replaced by any equivalent pair, the difference so introduced disappears on integration and summation.

Notice that we can transfer the terms (in the first bracket) evaluated at A_r to O, since

$$\begin{aligned}
&[\theta_r \psi_r(\xi_r \eta_r)V]^{A_r} \\
&\quad = [V\psi_r(\partial_x, \partial_y)U_r]^{A_r}, \quad \text{since } PA_r \text{ is a characteristic } C_r, \\
&\quad = [V\psi_r(\partial_x, \partial_y)U_r]^{O} + \int_O^{A_r} d\{V\psi_r(\partial_x, \partial_y)U_r\}. \qquad (35)
\end{aligned}$$

We may notice too that the data on the base-curve require the values of V and its derivatives up to and including those of order $m-1$. Actually, of course, it is sufficient to know V and one derivative of each such order, for the remaining derivatives can be obtained by differentiating along the base-curve.

If the base-curve is itself a characteristic (C_1, say), the argument breaks down, for two characteristics of the same suffix do not meet and so A_1 does not exist. This opens up possibilities with which I cannot at present cope and I postpone this case for further consideration.

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ON SOME BIORTHOGONAL SETS OF FUNCTIONS

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1. In this note I wish to point out that from every known orthogonal set of functions $\{\phi_n\}$ or from a biorthogonal set of functions $\{\chi_n, \phi_n\}$ new biorthogonal or, in exceptional cases, orthogonal sets can be formed by fractional integration by parts.

Let us take at first a finite interval (a, b). Then the formula of fractional integration by parts runs

$$\int_a^b f(x)\frac{d^\nu g(x)}{d(b-x)^\nu}\,dx = \int_a^b g(x)\frac{d^\nu f(x)}{d(x-a)^\nu}\,dx. \tag{1}$$

The fractional derivatives occurring in (1) can be defined by integrals if the real part of ν is negative. Thus

$$\frac{d^\nu f(x)}{d(x-a)^\nu} = \frac{1}{\Gamma(-\nu)}\int_a^x (x-y)^{-\nu-1}f(y)\,dy \quad [\Re(\nu) < 0] \tag{2}$$

and

$$\frac{d^\nu g(x)}{d(b-x)^\nu} = \frac{1}{\Gamma(-\nu)}\int_x^b (y-x)^{-\nu-1}g(y)\,dy \quad [\Re(\nu) < 0]. \tag{3}$$

With these definitions of the fractional derivatives, (1) was proved by Love and Young* for functions f and g belonging to classes $L^p(a, b)$ and $L^q(a, b)$ respectively† with the restrictions

$$p > 1, \qquad q > 1, \qquad -1 < \nu < 0, \qquad p^{-1}+q^{-1}+\nu \leqslant 1. \tag{4}$$

For other values of ν their Theorem C combined with ordinary integration by parts enables us to ascertain the validity of (1).

For many purposes, especially for transformation of hypergeometric integrals by means of fractional integration by parts, a very

* E. R. Love and L. C. Young, *Proc. London Math. Soc.* (2) 44 (1938), 1–34, Theorem C.

† For the definition of classes L^p see, for instance, E. C. Titchmarsh, *Theory of Functions* (Oxford, 1932), § 12.4.

much less deep and more restrictive proof of rule (1) is sufficient.* If f and g possess expansions

$$f(x) = \sum_{r=0}^{\infty} A_r(x-a)^{\rho+r-1}, \qquad g(x) = \sum_{s=0}^{\infty} B_s(b-x)^{\sigma+s-1}, \tag{5}$$

the radius of convergence of which exceeds $b-a$, and if the fractional derivatives are obtained by differentiating these series term by term and using the definition

$$\frac{d^\nu w^{\mu-1}}{dw^\nu} = \frac{\Gamma(\mu)w^{\mu-\nu-1}}{\Gamma(\mu-\nu)}, \tag{6}$$

then (1) is true if both integrals are convergent, i.e. when

$$\Re(\rho) > 0, \qquad \Re(\rho-\nu) > 0, \qquad \Re(\sigma) > 0, \qquad \Re(\sigma-\nu) > 0. \tag{7}$$

Obviously both definitions of fractional derivatives are in accordance in those cases in which both definitions have a meaning.

2. Now let us take a biorthogonal system $\{\chi_n, \phi_n\}$ $(n = 0, 1, 2,...)$ satisfying the relations

$$\int_a^b \phi_m(x)\chi_n(x)\,dx = c_n\,\delta_{mn}, \tag{8}$$

where δ_{mn} is Kronecker's delta. Define a new set of functions by

$$\Phi_n(x) = v(x)\frac{d^\nu}{d(x-a)^\nu}\{u(x)\phi_n(x)\} \tag{9}$$

and

$$\mathrm{X}_n(x) = \frac{1}{v(x)}\frac{d^{-\nu}}{d(b-x)^{-\nu}}\left\{\frac{\chi_n(x)}{u(x)}\right\}. \tag{10}$$

Then, under certain conditions, $\{\Phi_n, \mathrm{X}_n\}$ is a biorthogonal set, and

$$\int_a^b \Phi_m(x)\mathrm{X}_n(x)\,dx = c_n\,\delta_{mn}. \tag{11}$$

Here $u(x)$ and $v(x)$ must be chosen so as to fulfil the conditions of validity. Otherwise these functions are arbitrary.

The identity of (11) and (8) follows from (1) by putting

$$f(x) = u(x)\phi_m(x), \qquad g(x) = v(x)\mathrm{X}_n(x).$$

The validity of (11) follows then, for instance, from Theorem C of Love and Young, if $u(x)\phi_m(x)$ and $v(x)\mathrm{X}_n(x)$ belong to classes $L^p(a,b)$ and $L^q(a,b)$ respectively so that (4) holds. In this case $\Phi_n(x)$

* A. Erdélyi, *Quart. J. of Math.* (Oxford), 10 (1939), 129–34 and 176–89.

is defined by (9) and (2) directly; $\mathrm{X}_n(x)$ is defined from (10) and (3) as the solution of Abel's integral equation

$$\frac{\chi_n(x)}{u(x)} = \frac{1}{\Gamma(-\nu)} \int_x^b (y-x)^{-\nu-1} v(y) \mathrm{X}_n(y)\, dy,$$

provided that this integral equation has a solution and $v(y)\mathrm{X}_n(y)$ belongs to $L^q(a,b)$.

We can prove (11) the other way round, taking

$$f(x) = \Phi_m(x)/v(x), \qquad g(x) = \chi_n(x)/u(x)$$

and replacing ν by $-\nu$ in (1). Then the validity of (11) follows when $\Phi_m(x)/v(x)$ and $\chi_n(x)/u(x)$ belong to classes $L^p(a,b)$ and $L^q(a,b)$ respectively so that

$$p > 1, \qquad q > 1, \qquad 0 < \nu < 1, \qquad p^{-1}+q^{-1}-\nu \leqslant 1.$$

In this case $\mathrm{X}_n(x)$ is defined by (10) and (3) directly, whereas $\Phi_n(x)$ is the solution of Abel's integral equation

$$u(x)\phi_n(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-y)^{\nu-1} \frac{\Phi_n(y)}{v(y)}\, dy.$$

If the functions concerned have expansions of the type (5), then (7) is in most cases less restrictive than (4). However, it is not important to obtain the least restrictive conditions from the partial integration, for, at least in the examples considered below, by analytic continuation in ν and the other parameters involved, the validity of (11) in the widest possible range immediately follows when only its validity in some range of the parameters has been proved.

3. Completeness and closedness of the biorthogonal set $\{\mathrm{X}_n, \Phi_m\}$ can also be inferred from the corresponding properties of the set $\{\chi_n, \phi_n\}$.

To fix ideas, let us take $u(x) \equiv v(x) \equiv 1$ and suppose that $\{\phi_n\}$ is complete with respect to some $L^p(a,b)$ $(p > 1)$. Then $\{\Phi_n\}$ is complete* with respect to $L^q(a,b)$, where $q = p/(1-p\nu)$; here $-1 < \nu < 0$ is assumed.

* Dr. Kober adds that $\{\Phi_n\}$ is even complete with respect to any $L^r(a,b)$ where $r \geqslant q$. If $\nu \leqslant -1$, then $\{\Phi_n\}$ is complete with respect to any $L^r(a,b)$ with $r \geqslant 1$.

To prove this we have to show that from

$$\int_a^b \Phi_n(x)F(x)\,dx = 0$$

for every $n = 0, 1, 2,\ldots$, it follows that $F(x)$ is a null-function. Now

$$0 = \int_a^b \Phi_n(x)F(x)\,dx = \int_a^b \frac{d^\nu \phi_n(x)}{d(x-a)^\nu} F(x)\,dx = \int_a^b \phi_n(x)\frac{d^\nu F(x)}{d(b-x)^\nu}\,dx.$$

From $F \in L^q(a,b)$ by a known theorem of Hardy and Littlewood*

$$\frac{d^\nu F(x)}{d(b-x)^\nu} \in L^p(a,b), \qquad p = \frac{q}{1+q\nu}$$

follows. Hence, from the completeness of $\{\phi_n\}$ it follows that $d^\nu F(x)/d(b-x)^\nu$ is zero almost everywhere, and so is $F(x)$. This last step follows, for instance, from a theorem of Tamarkin.†

The completeness of $\{X_n\}$ cannot be inferred from that of $\{\chi_n\}$. This is of little consequence, however, for the completeness of the *biorthogonal* system $\{X_n, \Phi_n\}$ depends only‡ on the completeness of $\{\Phi_n\}$.

4. As an example Jacobi's polynomials may be taken. Then (8) certainly holds with§ $a = 0$, $b = 1$,

$$\phi_n(x) = F(-n, \alpha+n; \gamma; x), \tag{12}$$

$$\begin{aligned}\chi_n(x) &= \frac{\Gamma(\alpha-\gamma+1)\Gamma(\gamma+n)}{\Gamma(\alpha-\gamma+n+1)\Gamma(\gamma)} x^{\gamma-1}(1-x)^{\alpha-\gamma}F(-n, \alpha+n; \gamma; x) \qquad (13)\\ &= (-)^n x^{\gamma-1}(1-x)^{\alpha-\gamma}F(-n, \alpha+n; \alpha-\gamma+1; 1-x)\\ &= (-)^n(1-x)^{\alpha-\gamma}F(\alpha-\gamma+n+1, 1-\gamma-n; \alpha-\gamma+1; 1-x),\end{aligned}$$

and

$$c_n = \frac{n!\Gamma(\gamma)\Gamma(\alpha-\gamma+1)}{(\alpha+2n)\Gamma(\alpha+n)}, \tag{14}$$

provided that $\Re(\alpha)+1 > \Re(\gamma) > 0$.

* G. H. Hardy and J. E. Littlewood, *Math. Zeits.* 27 (1928), 565–606, Theorem 4.

† J. D. Tamarkin, *Annals of Math.* 31 (1930), 219–29, Theorem 4.

‡ S. Kaczmarz und H. Steinhaus, *Theorie der Orthogonalreihen* (Warszawa-Lwów, 1935), 262 [814].

§ Cf. e.g. P. Appell et J. Kampé de Fériet, *Fonctions hypergéométriques. Polynomes d'Hermite* (Paris, 1926), p. 11. In (13) known transformation formulae of the hypergeometric series have been used. See E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge, 1927), §§ 14.53 and 14.4.

At first let us take in the definition of Φ_n and X_n

$$\nu = 1-\lambda, \qquad u(x) \equiv v(x) \equiv 1.$$

Then we have* according to definition (5), (6)

$$\Phi_n(x) = \frac{d^{1-\lambda}}{dx^{1-\lambda}}\{F(-n, \alpha+n; \gamma; x)\} = \frac{x^{\lambda-1}}{\Gamma(\lambda)}\,{}_3F_2(-n, \alpha+n, 1; \gamma, \lambda; x). \tag{15}$$

Again, according to the same definition,†

$$\begin{aligned} X_n(x) &= \frac{d^{\lambda-1}}{d(1-x)^{\lambda-1}}\{(-)^n(1-x)^{\alpha-\gamma}\times \\ &\qquad \times F(\alpha-\gamma+n+1, 1-\gamma-n; \alpha-\gamma+1; 1-x)\} \\ &= \frac{(-)^n\Gamma(\alpha-\gamma+1)}{\Gamma(\alpha-\gamma-\lambda+2)}(1-x)^{\alpha-\gamma-\lambda+1}\times \\ &\qquad \times F(\alpha-\gamma+n+1, 1-\gamma-n; \alpha-\gamma-\lambda+2; 1-x). \end{aligned} \tag{16}$$

Term-by-term fractional differentiation has to be justified in (16), the radius of convergence being only unity. This justification is easily supplied provided that

$$\Re(\gamma) > 0, \qquad \Re(\lambda) > 0, \qquad \Re(\alpha-\gamma-\lambda+2) > 0. \tag{17}$$

Although this derivation of the orthogonality relation

$$\begin{aligned} &\int_0^1 x^{\lambda-1}(1-x)^{\alpha-\gamma-\lambda+1}\,{}_3F_2(-m, \alpha+m, 1; \gamma, \lambda; x)\times \\ &\qquad \times F(\alpha-\gamma+n+1, 1-\gamma-n; \alpha-\gamma-\lambda+2; 1-x)\,dx \\ &\qquad\qquad = \frac{(-)^n n!\Gamma(\gamma)\Gamma(\lambda)\Gamma(\alpha-\gamma-\lambda+2)}{(\alpha+2n)\Gamma(\alpha+n)}\delta_{mn} \end{aligned} \tag{18}$$

is only valid when, besides (17), $\Re(\alpha)+1 > \Re(\gamma)$ also holds, by analytical continuation the validity of (18) follows with only the restrictions (17).

The set given by (15) and (16) is essentially identical with a biorthogonal system discussed by Ignatowsky.‡

5. From (12) and (13) a more general biorthogonal set can be obtained by putting

$$\nu = \beta-\lambda, \qquad u(x) = x^{\beta-1}(1-x)^{-\epsilon}, \qquad v(x) \equiv 1.$$

* A. Erdélyi, *Quart. J. of Math.* (Oxford), 10 (1939), 176–89, (184).

† Ibid., 178.

‡ W. v. Ignatowsky, *Comptes Rendus de l'Acad. de U.S.S.R.* (1931), 179–85 and 186–9.

Then,

$$\Phi_n(x) = \frac{d^{\beta-\lambda}}{dx^{\beta-\lambda}}\{x^{\beta-1}(1-x)^{-\epsilon}F(-n, \alpha+n; \gamma; x)\}$$

$$= \frac{d^{\beta-\lambda}}{dx^{\beta-\lambda}}\left\{\sum_{r=0}^{n}\sum_{s=0}^{\infty}\frac{(-n)_r(\alpha+n)_r}{r!\,(\gamma)_r}\frac{(\epsilon)_s}{s!}x^{\beta+r+s-1}\right\}$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{\infty}\frac{\Gamma(\beta+r+s)}{\Gamma(\lambda+r+s)}\frac{(-n)_r(\alpha+n)_r}{r!\,(\gamma)_r}\frac{(\epsilon)_s}{s!}x^{\lambda+r+s-1}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\lambda)}x^{\lambda-1}\sum_{r=0}^{n}\sum_{s=0}^{\infty}\frac{(\beta)_{r+s}}{(\lambda)_{r+s}}\frac{(-n)_r(\alpha+n)_r}{r!\,(\gamma)_r}\frac{(\epsilon)_s}{s!}x^{r+s}, \tag{19}$$

and similarly, using the second form of $\chi_n(x)$,

$$\mathrm{X}_n(x) = \frac{d^{\lambda-\beta}}{d(1-x)^{\lambda-\beta}}\{(-)^n[1-(1-x)]^{\gamma-\beta}(1-x)^{\alpha-\gamma+\epsilon}\times$$

$$\times F(-n, \alpha+n; \alpha-\gamma+1; 1-x)\}$$

$$= \frac{(-)^n\Gamma(\alpha-\gamma+\epsilon+1)}{\Gamma(\alpha+\beta-\gamma+\epsilon-\lambda+1)}(1-x)^{\alpha+\beta-\gamma+\epsilon-\lambda}\times$$

$$\times\sum_{r=0}^{n}\sum_{s=0}^{\infty}\frac{(\alpha-\gamma+\epsilon+1)_{r+s}}{(\alpha+\beta-\gamma+\epsilon-\lambda+1)_{r+s}}\frac{(-n)_r(\alpha+n)_r}{r!\,(\alpha-\gamma+1)_r}\frac{(\beta-\gamma)_s}{s!}(1-x)^{r+s}. \tag{20}$$

Both Φ_n and X_n are expressible in this case in terms of hypergeometric series of two variables. In the notation of Appell and Kampé de Fériet (loc. cit., p. 150)

$$\Phi_n(x) = \frac{\Gamma(\beta)}{\Gamma(\lambda)}x^{\lambda-1}F\left(\begin{matrix}1 & \beta \\ 2 & -n, \epsilon; \alpha+n, \eta \\ 1 & \lambda \\ 1 & \gamma, \eta\end{matrix}\middle| x, x\right), \tag{21}$$

and

$$\mathrm{X}_n(x) = \frac{(-)^n\Gamma(\alpha-\gamma+\epsilon+1)}{\Gamma(\alpha+\beta-\gamma+\epsilon-\lambda+1)}(1-x)^{\alpha+\beta-\gamma+\epsilon-\lambda}\times$$

$$\times F\left(\begin{matrix}1 & \alpha-\gamma+\epsilon+1 \\ 2 & -n, \beta-\gamma; \alpha+n, \eta \\ 1 & \alpha+\beta-\gamma+\epsilon-\lambda+1 \\ 1 & \alpha-\gamma+1, \eta\end{matrix}\middle| 1-x, 1-x\right). \tag{22}$$

Here η is a 'dummy' parameter.

From known properties of hypergeometric functions we easily obtain

$$\begin{aligned}
\Phi_n(x) &= O(x^{\lambda-1}) && (x \to 0),\\
\Phi_n(x) &= O\{(1-x)^{\lambda-\beta-\epsilon}\}+O(1) && (x \to 1),\\
X_n(x) &= O(x^{\gamma-\lambda})+O(1) && (x \to 0),\\
X_n(x) &= O\{(1-x)^{\alpha+\beta-\gamma+\epsilon-\lambda}\} && (x \to 1).
\end{aligned}$$

Hence (11) is valid in this case, by the theory of analytical continuation, with only the restrictions

$$\Re(\alpha)+1 > \Re(\gamma) > 0, \qquad \Re(\alpha+\beta-\gamma+\epsilon)+1 > \Re(\lambda) > 0. \quad (23)$$

In certain special cases, for instance when $\beta = 1$, $\epsilon = 0$, which leads us to Ignatowsky's system, some of the restrictions can be dispensed with.

6. Now let us proceed to an infinite interval. Without loss of generality we may take it to be $(0, \infty)$. Here the rule of fractional integration by parts runs

$$\int_0^\infty f(x)\frac{d^\nu g(x)}{d(-x)^\nu}\,dx = \int_0^\infty g(x)\frac{d^\nu f(x)}{dx^\nu}\,dx. \quad (24)$$

The corresponding definitions for the fractional derivatives are

$$\frac{d^\nu f(x)}{dx^\nu} = \frac{1}{\Gamma(-\nu)}\int_0^x (x-y)^{-\nu-1}f(y)\,dy \quad [\Re(\nu) < 0] \quad (25)$$

and

$$\frac{d^\nu g(x)}{d(-x)^\nu} = \frac{1}{\Gamma(-\nu)}\int_x^\infty (y-x)^{-\nu-1}g(y)\,dy \quad [\Re(\nu) < 0]. \quad (26)$$

So far as I am aware, no proof of (24) has been published yet. Dr. H. Kober, however, writes to me* that he has proved (24), e.g. for functions $f(x)$ and $x^{-\nu}g(x)$ belonging to classes $L^p(0,\infty)$ and $L^q(0,\infty)$ respectively, so that

$$1 < p \leqslant \infty, \qquad q = p/(p-1). \quad (27)$$

It is quite clear that in this case again under suitable conditions expansion and formal differentiation may be used instead of (25), (26). Again, $f(x)$ is to be expanded into a series of the type (5) with

* This is only a particular case of the results contained in Dr. Kober's forthcoming paper 'On fractional integrals and derivatives' (Theorem 3). For $1 < p < \infty$ the result quoted here can be deduced from Theorem 329 of *Inequalities* by G. H. Hardy, J. E. Littlewood, and G. Pólya (Cambridge, 1934).

$a = 0$; $g(x)$, however, into a series of exponential functions, the derivatives of which are defined by

$$\frac{d^\nu e^{-kx}}{d(-x)^\nu} = k^\nu e^{-kx}. \tag{28}$$

Now, again, (11) can be derived from (8), by taking $a = 0$, $b = \infty$ in both equations. By taking

$$\Re(\nu) < 0, \qquad f(x) = u(x)\phi_m(x), \qquad g(x) = v(x)\mathrm{X}_n(x)$$

in (24), the validity of (11) follows from Kober's theorem if $u(x)\phi_m(x)$ and $x^{-\nu}v(x)\mathrm{X}_n(x)$ belong to classes $L^p(0,\infty)$ and $L^q(0,\infty)$ respectively, so that (27) holds. Φ_n is then defined directly by (25), whereas X_n is the solution of the integral equation

$$\frac{\chi_n(x)}{u(x)} = \frac{1}{\Gamma(-\nu)} \int_x^\infty (y-x)^{-\nu-1} v(y)\mathrm{X}_n(y)\, dy.$$

This integral equation reduces by the substitutions $x^{-1} = \xi$, $y^{-1} = \eta$ to the usual form of Abel's integral equation.

From Kober's theorem we can also prove (11) for $\Re(\nu) > 0$ if $\Phi_m(x)/v(x)$ and $x^\nu\chi_n(x)/u(x)$ belong to classes $L^p(0,\infty)$ and $L^q(0,\infty)$ respectively, so that (27) holds.

Completeness and closedness of $\{\mathrm{X}_n, \Phi_n\}$ can again be inferred from the corresponding properties of $\{\chi_n, \phi_n\}$.

7. As an example of a set in an infinite interval Laguerre polynomials may be taken. In order to do so we may put

$$\phi_n(x) = L_n^{(\alpha)}(2x), \tag{29}$$

$$\chi_n(x) = e^{-2x}x^\alpha L_n^{(\alpha)}(2x), \tag{30}$$

$$c_n = \frac{\Gamma(\alpha+n+1)}{n!\,2^{\alpha+1}}. \tag{31}$$

To begin with, we put

$$u(x) = e^{-x}, \qquad v(x) = x^{\frac{1}{2}\alpha+\nu}.$$

Then

$$\Phi_n(x) = x^{\frac{1}{2}\alpha+\nu}\frac{d^\nu}{dx^\nu}\{e^{-x}L_n^{(\alpha)}(2x)\} \tag{32}$$

and

$$\mathrm{X}_n(x) = x^{-\frac{1}{2}\alpha-\nu}\frac{d^{-\nu}}{d(-x)^{-\nu}}\{e^{-x}x^\alpha L_n^{(\alpha)}(2x)\}. \tag{33}$$

Φ_n and X_n in this case can be expressed in terms of confluent hypergeometric functions. For

$$\frac{d^\nu}{dx^\nu}\{e^{-x}x^r\} = \frac{e^{-x}}{\Gamma(-\nu)}\int_0^x (x-y)^{-\nu-1}y^r e^{x-y}\,dy$$

$$= \frac{e^{-x}}{\Gamma(-\nu)}\sum_{m=0}^{\infty}\frac{1}{m!}\int_0^x (x-y)^{m-\nu-1}y^r\,dy$$

$$= \frac{r!\,e^{-x}}{\Gamma(-\nu)}\sum_{m=0}^{\infty}\frac{\Gamma(m-\nu)x^{m-\nu+r}}{m!\Gamma(m-\nu+r+1)}$$

$$= \frac{r!\,x^{r-\nu}e^{-x}}{\Gamma(r-\nu+1)}\,{}_1F_1(-\nu;r-\nu+1;x)$$

and
$$L_n^{(\alpha)}(2x) = \sum_{r=0}^{n}\binom{n+\alpha}{n-r}\frac{(-2x)^r}{r!}.$$

Hence

$$\Phi_n(x) = x^{\frac{1}{2}\alpha+\nu}\sum_{r=0}^{n}\binom{n+\alpha}{n-r}\frac{(-2)^r}{r!}\frac{d^\nu}{dx^\nu}\{e^{-x}x^r\}$$

$$= x^{\frac{1}{2}\alpha}e^{-x}\sum_{r=0}^{n}\binom{n+\alpha}{n-r}\frac{(-2x)^r}{\Gamma(r-\nu+1)}\,{}_1F_1(-\nu;r-\nu+1;x). \quad (34)$$

If $\nu = 0, 1, 2,\ldots$, the confluent hypergeometric functions occurring in (34) reduce to polynomials.

Again*

$$\frac{d^{-\nu}}{d(-x)^{-\nu}}\{e^{-x}x^{\alpha+r}\} = \frac{e^{-x}}{\Gamma(\nu)}\int_x^\infty (y-x)^{\nu-1}y^{\alpha+r}e^{-(y-x)}\,dy$$

$$= e^{-x}x^{\alpha+r}\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-t}t^{\nu-1}\left(1+\frac{t}{x}\right)^{\alpha+r}dt$$

$$= e^{-x}x^{\alpha+r}\,{}_2F_0\left(\nu, -\alpha-r; -\frac{1}{x}\right),$$

when we define the divergent hypergeometric series ${}_2F_0$ uniquely by†

$$W_{k,m}(z) = e^{-\frac{1}{2}z}z^k\,{}_2F_0\left(\tfrac{1}{2}-k-m, \tfrac{1}{2}-k+m; -\frac{1}{z}\right). \quad (35)$$

* Cf. *Modern Analysis*, § 16.12. † Cf. also *Modern Analysis*, § 16.3.

Hence

$$X_n(x) = x^{-\frac{1}{2}\alpha-\nu} \sum_{r=0}^{n} \binom{n+\alpha}{n-r} \frac{(-2)^r}{r!} \frac{d^{-\nu}}{d(-x)^{-\nu}} \{e^{-x} x^{\alpha+r}\}$$

$$= x^{\frac{1}{2}\alpha-\nu} e^{-x} \sum_{r=0}^{n} \binom{n+\alpha}{n-r} \frac{(-2x)^r}{r!} {}_2F_0\left(\nu, -\alpha-r; -\frac{1}{x}\right). \qquad (36)$$

The sum in this reduces to a polynomial if $\nu = 0, -1, -2, \ldots$ or if $\alpha = 1, 2, \ldots$.

From the theory of the confluent hypergeometric functions it easily follows that

$$\Phi_n(x) = O(x^{\frac{1}{2}\alpha}) \qquad (x \to 0),$$

$$\Phi_n(x) = O(x^{\frac{1}{2}\alpha-1}) \qquad (x \to \infty),$$

$$X_n(x) = O(x^{\frac{1}{2}\alpha}) + O(x^{-\frac{1}{2}\alpha-\nu}) \qquad (x \to 0),$$

$$X_n(x) = O(x^{\frac{1}{2}\alpha-\nu} e^{-x}) \qquad (x \to \infty).$$

From this, by Kober's theorem with $p = \infty$ and $q = 1$, the permissibility of fractional integration by parts under the assumptions

$$\Re(\alpha) > -1, \qquad \Re(\nu) < 1$$

readily follows.

8. The biorthogonal system of the preceding section has been discussed in some detail because a particular case of it is of some importance in the theory of 'cut' Hankel transforms.

Put in the formulae of the preceding section

$$\alpha = \mu + 2l, \qquad \nu = -l, \qquad l = 1, 2, 3, \ldots.$$

Then our biorthogonal system is

$$\Phi_n(x) = \frac{x^{\frac{1}{2}\mu}}{(l-1)!} \int_0^x (x-y)^{l-1} e^{-y} L_n^{(\mu+2l)}(2y)\, dy, \qquad (37)$$

$$X_n(x) = (-)^l x^{-\frac{1}{2}\mu} \frac{d^l}{dx^l} \{e^{-x} x^{\mu+2l} L_n^{(\mu+2l)}(2x)\}, \qquad (38)$$

and the corresponding relation of orthogonality

$$\int_0^\infty \Phi_m(x) X_n(x)\, dx = \frac{\Gamma(\mu+n+2l+1)}{n!\, 2^{\mu+2l+1}} \delta_{mn}. \qquad (39)$$

It is readily shown from (38) and also easily seen from (36) that in this case $X_n(x)$ reduces to $x^{\frac{1}{2}\mu+\lambda} e^{-x} \times$ a polynomial of degree $n+l$ in x; (39) is convergent when $\Re(\mu)+2l+1 > 0$.

The significance of these functions comes from the fact that the biorthogonal system $\{X_n, \Phi_n\}$ is with respect to $L^2(0,\infty)$ a complete set of self and skew reciprocal functions of the cut Hankel transform,* i.e.

$$\left.\begin{matrix}\Phi_n(x)\\ X_n(x)\end{matrix}\right\} = (-)^n \int_0^\infty J_{\mu,l}\{2\sqrt{(xy)}\} \left\{\begin{matrix}\Phi_n(y)\\ X_n(y)\end{matrix}\right\} dy \tag{40}$$

$$[-1 < \Re(\mu)+2l < 1,\ l = 1,\ 2,\ 3,\ldots].$$

Here $J_{\mu,l}$ denotes the cut Bessel function of first kind

$$J_{\mu,l}(2\sqrt{z}) = \sum_{r=l}^{\infty} \frac{(-)^r z^{\frac{1}{2}\mu+r}}{r!\Gamma(\mu+r+1)}. \tag{41}$$

Any function the expansion of which into a series of the Φ_n (or, alternatively, the X_n) contains only functions of even (odd) index n is self (skew) reciprocal in the cut Hankel transform.

The proof of (40) and a few more properties of the biorthogonal set $\{X_n, \Phi_n\}$ will be given elsewhere.

9. The biorthogonal sets dealt with in the last two sections are by no means the only ones which can be derived from Laguerre polynomials by partial integration. Leaving to the reader the discussion of a more general case which, like that dealt with in § 5, leads to functions expressible in terms of confluent hypergeometric functions of *two* variables, I confine myself to another simple example connected with certain expansions in series of Whittaker functions $W_{k,m}$ or, what is the same thing, confluent hypergeometric series ${}_2F_0$.

Let us again take (29), (30), but now with $u(x) \equiv 1$, $v(x) = x^\nu e^{-x}$. Then we have

$$\Phi_n(x) = e^{-x}x^\nu \frac{d^\nu}{dx^\nu}\{L_n^{(\alpha)}(2x)\} = e^{-x}x^\nu \sum_{r=0}^{n} \binom{n+\alpha}{n-r} \frac{(-2)^r}{r!} \frac{d^\nu x^r}{dx^\nu}$$

$$= e^{-x} \sum_{r=0}^{n} \binom{n+\alpha}{n-r} \frac{(-2x)^r}{\Gamma(r-\nu+1)}.$$

Thus

$$\Phi_n(x) = \frac{\Gamma(n+\alpha+1)e^{-x}}{n!\Gamma(1+\alpha)\Gamma(1-\nu)} {}_2F_2(-n, 1; 1+\alpha, 1-\nu; 2x). \tag{42}$$

* The cut Hankel transform is mentioned in E. C. Titchmarsh's *Theory of Fourier Integrals* (Oxford, 1937), § 8.4, Example (1); a full discussion of cut Hankel transforms was given by H. Kober, *Quart. J. of Math.* (Oxford), 8 (1937), 186–99. Kober, in a paper which will appear in the *Proceedings of the Edinburgh Mathematical Society*, has also given a complete system of self and skew reciprocal functions of the cut Hankel transform.

Again,

$$X_n(x) = x^{-\nu}e^x \frac{d^{-\nu}}{d(-x)^{-\nu}}\{e^{-2x}x^\alpha L_n^{(\alpha)}(2x)\}$$

$$= \frac{x^{-\nu}e^x}{\Gamma(\nu)} \int_x^\infty (y-x)^{\nu-1}e^{-2y}y^\alpha L_n^{(\alpha)}(2y)\, dy.$$

Putting $y = x+\frac{1}{2}t$, this becomes

$$\frac{x^{\alpha-\nu}e^{-x}}{2^\nu\Gamma(\nu)} \int_0^\infty e^{-t}t^{\nu-1}L_n^{(\alpha)}(2x+t)\left(1+\frac{t}{2x}\right)^\alpha dt$$

$$= \frac{(-1)^n}{n!}\, 2^{-\frac{1}{2}(\nu+\alpha+1)}x^{\frac{1}{2}(\alpha-\nu-1)}W_{\frac{1}{2}(\alpha-\nu+1)+n,\frac{1}{2}(\alpha+\nu)}(2x),$$

by an integral representation of Whittaker's function which I gave in a recent paper.* In consequence of (35) this can also be written

$$X_n(x) = \frac{(-1)^n}{n!}\, 2^{n-\nu}e^{-x}x^{\alpha+n-\nu}\, {}_2F_0\left(-n-\alpha, \nu-n; -\frac{1}{2x}\right). \qquad (43)$$

It is easily shown that, in general,

$$\Phi_n(x) = O(1) \quad (x \to 0), \qquad \Phi_n(x) = O(x^n e^{-x}) \quad (x \to \infty),$$

$$X_n(x) = O(x^\alpha)+O(x^{-\nu}) \quad (x \to 0), \qquad X_n(x) = O(x^{\alpha+n-\nu}e^{-x}) \quad (x \to \infty).$$

Hence

$$\int_0^\infty e^{-2x}x^{\alpha+n-\nu}\, {}_2F_2(-m, 1; \alpha+1, 1-\nu; 2x)\, {}_2F_0\left(-\alpha-n, \nu-n; -\frac{1}{2x}\right) dx$$

$$= (-)^n 2^{\nu-n-\alpha-1}n!\Gamma(\alpha+1)\Gamma(1-\nu)\delta_{mn} \qquad (44)$$

is absolutely convergent whenever $\Re(\alpha) > -1$, $\Re(\nu) < 1$.

10. The most interesting particular case of the system (42), (43) arises when $-\nu$ is a non-negative integer, l say. In this case ${}_2F_0$ in (43) is a terminating hypergeometric series, and both Φ_n and X_n can be expressed in terms of Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!}\frac{d^n(e^{-x}x^{\alpha+n})}{dx^n} = \sum_{r=0}^n \binom{n+\alpha}{n-r}\frac{(-x)^r}{r!} \qquad (45)$$

and 'cut' Laguerre polynomials

$$L_{n,l}^{(\alpha)}(x) = \sum_{r=l}^n \binom{n+\alpha}{n-r}\frac{(-x)^r}{r!} \quad (n \geqslant l) \qquad (46)$$

which are obtained by omitting the first l terms in $L_n^{(\alpha)}(x)$.

* A. Erdélyi, *Proc. of Benares Math. Soc.* 1 (1939), 39–53, equation (23).

Instead of reducing Φ_n and X_n of the last section in the particular case $\nu = -l$ ($l = 0, 1, 2,\ldots$) to the polynomials mentioned above, I conclude by proving directly by ordinary partial integration that the functions

$$f_n(x) = x^{\frac{1}{2}(\alpha+l)}e^{-\frac{1}{2}x}L_{n+l}^{(\alpha)}(x) \quad (n = 0, 1, 2,\ldots) \tag{47}$$

and

$$g_n(x) = x^{\frac{1}{2}(\alpha-l)}e^{-\frac{1}{2}x}L_{n+l,l}^{(\alpha)}(x) \quad (n = 0, 1, 2,\ldots) \tag{48}$$

form a biorthogonal system if $\Re(\alpha) > -l-1$.

Plainly, both $f_m(x)$ and $g_n(x)$ belong to $L^2(0,\infty)$ if $\Re(\alpha) > -l-1$, and hence the integral

$$\int_0^\infty f_m g_n\,dx = \int_0^\infty e^{-x}x^\alpha L_{m+l}^{(\alpha)}(x)L_{n+l,l}^{(\alpha)}(x)\,dx \tag{49}$$

is absolutely convergent. Now, from (45),

$$(m+l)!\,e^{-x}x^\alpha L_{m+l}^{(\alpha)}(x) = \frac{d^{m+l}}{dx^{m+l}}(e^{-x}x^{\alpha+m+l})$$

$$= \frac{d^l}{dx^l}\left\{\frac{d^m(e^{-x}x^{\alpha+m+l})}{dx^m}\right\} = \frac{d^l}{dx^l}\{m!\,e^{-x}x^{\alpha+l}L_m^{(\alpha+l)}(x)\}.$$

Hence

$$\int_0^\infty f_m g_n\,dx = \frac{m!}{(m+l)!}\int_0^\infty \frac{d^l}{dx^l}\{e^{-x}x^{\alpha+l}L_m^{(\alpha+l)}(x)\}L_{n+l,l}^{(\alpha)}(x)\,dx$$

$$= \frac{(-1)^l m!}{(m+l)!}\int_0^\infty e^{-x}x^{\alpha+l}L_m^{(\alpha+l)}(x)\frac{d^l}{dx^l}\{L_{n+l,l}^{(\alpha)}(x)\}\,dx,$$

by integrating l times by parts, since the integrated terms cancel. From (46) it is easily seen that

$$\frac{dL_{n,l}^{(\alpha)}(x)}{dx} = -L_{n-1,l-1}^{(\alpha+1)}(x)$$

and thus

$$\frac{d^l}{dx^l}\{L_{n+l,l}^{(\alpha)}(x)\} = (-1)^l L_n^{(\alpha+l)}(x).$$

Hence

$$\int_0^\infty f_m g_n\,dx = \frac{m!}{(m+l)!}\int_0^\infty e^{-x}x^{\alpha+l}L_m^{(\alpha+l)}(x)L_n^{(\alpha+l)}(x)\,dx.$$

The last integral being known to vanish unless $m = n$, and to be equal to $\Gamma(\alpha+l+n+1)/n!$ if $m = n$, we get finally

$$\int_0^\infty f_m g_n\,dx = \frac{\Gamma(\alpha+l+n+1)}{(n+l)!}\delta_{mn} \tag{50}$$

$$[\Re(\alpha) > -l-1;\ l, m, n = 0, 1, 2,\ldots].$$

I am indebted to Dr. Kober for many valuable suggestions.

KINEMATICAL RELATIVITY AND TEXTILE GEOMETRY

By H. G. FORDER (*Auckland*)

[Received 12 February 1940]

THE connexion between these two subjects has not, so far as I know, been previously noted.*

1. Consider any set of particle observers who can send signals to one another. We suppose that the observers form a one-dimensional series, and that, if signals be sent in both directions from any observer, one or other of these signals eventually reaches any specified observer. We do not assume at present any method of measuring distances between observers, or of assigning dates to the times of transmission of signals, but we can nevertheless picture the argument by representing the varying positions of the particles and the tracks of the signals by curves in a plane. The plane and curves need not form continua; it is sufficient if they contain the points we need and satisfy the usual two-dimensional order relations.

We suppose that two particles never collide, and that one signal never overtakes another. For convenience we suppose that any particle required is present, and that any signal required is sent. Thus through each point of our picture-plane we have three curves: (i) a 'path curve' representing the path of the particle, (ii) two signal curves representing the paths of the signals in the two directions. The path curves never meet, and signal curves of the same family never meet. Thus we have a 3-web in the sense of textile plane geometry.

2. Our first problem is to assign a 'date' to each event, i.e. to each transmission or reception of a signal. We shall assume that, given two positions of a particle P_1, there is another particle P_2 such that, if a signal is sent from P_1 in its first position and reflected from P_2, it reaches P_1 in its second position; and that, if a signal is sent from P_1 in the opposite direction, there is another particle P_3 which

* On kinematical relativity see in particular Milne, *Relativity, Gravitation, and World-structure* (Oxford, 1935); Milne and Whitrow, *Zeits. für Astrophys.* 15 (1938), 263; on textile geometry, Blaschke und Bol, *Geometrie der Gewebe* (Berlin, 1938), and the papers there quoted; on the related questions of foundations see Forder, *Foundations of Euclidean Geometry* (Cambridge, 1927), and Reidemeister, *Grundlagen der Geometrie* (Berlin, 1930).

serves the same purpose as P_2. The corresponding curves are seen in one half of Fig. 1.

We now affix dates or labels to the times of transmission of signals from P_1 and the times of reception of signals at P_1. We do not assume that these dates are numbers. Consider three positions of P_1 with dates u, t_1, t_2. By our assumption, there is a particle P_2 such that a signal from P_1 at date u after reflection at P_2 reaches P_1 again at date t_1, and there is another particle P_3 with that property. If, then, the signal sent from P_1 at date t_2 after reflection at P_2 reaches P_1 again at date t, we write $t_1+t_2=t$, where the plus sign denotes at present merely a functional relation. We shall assume that the signal from P_1 at t_2 after reflection at P_3 reaches P_1 also at date t. This is indicated in Fig. 1. We have thus a definition of the 'addition' of dates with respect to u, which takes the part of zero.

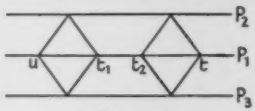

FIG. 1

We wish ultimately to compare our dates with real numbers, so that we naturally desire that the stretch of time from u to t_2 should 'equal' that from t_1 to t_1+t_2. We therefore assume at present that, if a signal from P_1 at date u after reflection at P_4 returns to P_1 at date t_2, then a signal from P_1 at t_1 after reflection at P_4 returns to P_1 at date t_1+t_2. This assumption is represented by Fig. 2. The same figure shows $t_2+t_1=t$, so that $t_1+t_2=t_2+t_1$. Thus addition of dates is commutative. The associative law will follow if Fig. 3 always closes, as is easily seen. Thomson* has shown that the closure of Fig. 3 follows from that of Fig. 2. Thus our assumptions already secure that the addition of dates satisfies the usual laws. So far, observer P_1 has been in a privileged position; other observers can date their events in a similar way under similar assumptions, but there is no correlation between the dates assigned by different observers.

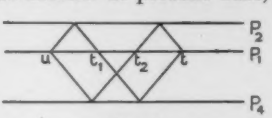

FIG. 2

FIG. 3

3. We now assume that the dates for any one particle form an ordered set, and that, if t_1, t_2 succeed u, then t_1+t_2 succeeds t_1 and t_2. We can define subtraction, and regard dates which succeed u as positive, those which precede u as negative. We define $2t$ as $t+t$, and can introduce $\frac{1}{2}t$, and any rational multiple or submultiple of t,

* Reidemeister, loc. cit. 78.

when we assume the existence of the necessary reflecting observers. Finally, if we assume the axiom of Archimedes and Dedekind continuity for our dates, we can set up a one-to-one correspondence between our dates and real numbers, in which addition corresponds to addition.

This correspondence is now to be set up along each path curve. Now suppose the points on the various path curves to which the same dates have been assigned are joined by a curve. Such curves do not intersect; through each point goes one such curve. The four families of curves we now have constitute a 4-web. We represent the new 'date-curves' by dotted lines.

4. Milne calls two observers P_1, P_2 'kinematically equivalent' when they can so date their events that, if a signal from P_1 at date t_1 reaches P_2 at t_2' (date assigned by P_2), and a signal from P_2 at t_1' reaches P_1 at t_2, then t_2' is the same function of t_1 that t_2 is of t_1'. He shows that two observers can always date their events so as to be kinematically equivalent, and considers the conditions which must hold if any number of observers are kinematically equivalent in pairs.

A special case of Milne's definition is this: P_1, P_2 are equivalent, if signals sent from P_1 to P_2 and from P_2 to P_1 at the same (local) dates reach P_2, P_1 respectively at the same dates. That is, we only assume that, if $t_1 = t_1'$, then $t_2 = t_2'$. This assumption is represented in Fig. 4.

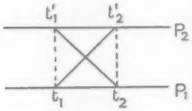

FIG. 4

If all our observers are equivalent in pairs, figures like Fig. 4 always close, and then by a theorem of Blaschke* the curves are topologically equivalent to four sets of parallel lines constituting the sides of parallelograms and their diagonals. Hence all systems of observers equivalent in pairs give rise to 4-webs topologically equivalent.

We could use the date curves to define addition of dates, $t = t_1 + t_2$,

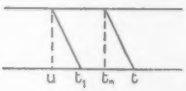

FIG. 5

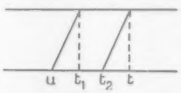

FIG. 6

as in Figs. 5, 6. These definitions will be equivalent to one another and to that given earlier, when Fig. 4 closes.

* *Math. Zeits.* 28 (1927).

Finally, if our observers are equivalent in pairs, then Fig. 7 always closes.

Now, if we assume at once that our dates are in one-to-one correspondence with real numbers, then by Blaschke (loc. cit.) from the closure of Fig. 7 we can deduce the closures of Figs. 2, 3, and, if we add the date-curves in obvious fashion, all the above consequences follow.

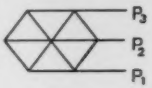

FIG. 7

The assumption corresponding to Fig. 7 can be stated as follows: if particle P_2 be between particles P_1 and P_3, and if signals sent in contrary ways from P_2 at one instant return to P_2 after reflection at P_1, P_3 respectively at the same instant, then, if at the instant these return, two signals are again sent from P_2 in contrary ways, then after reflection at P_1, P_3 respectively these also return to P_2 at the same instant.

When this holds for each case, and dates are real numbers, our observers form a system of equivalent observers.

5. If in Fig. 1 we define t as $t_1 t_2$, much of the above argument can be repeated with multiplication replacing addition. Instead of negative dates $-t$, $-2t$,... we now have t^{-1}, t^{-2},.... If we still wish for correspondence of dates and real numbers, which now must all be positive, we must assume that t^{-n} tends to a limit as n tends to infinity. This limit event must lie on all path curves, which therefore must proceed from a common point at a common date. The axiom of Archimedes prevents them from meeting again. When we took addition as basic, we could construct multiplication from it, as in the real number system. But with the present point of view the way from multiplication to addition is not so obvious.

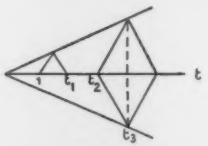

FIG. 8

If we introduce addition as indicated in Fig. 8, $t_3 = \frac{1}{2}(t_2+t)$, we can reach the relativity transformation, but as the work would only be a transcription of Milne's, it may be omitted.

The two ways of dating clearly correspond to the two measures of time which Milne has introduced.

It may perhaps be remarked that the figure corresponding to Fig. 2 in the present way of measuring time is a special case of Brianchon's theorem, which is thus connected with the commutative law of multiplication in the same way as Pappus's theorem has long been known to be.

ON EXPANSIONS IN EIGENFUNCTIONS (II)

By E. C. TITCHMARSH (*Oxford*)

[Received 2 December 1939]

1. In the previous paper* I showed that, if L denotes a certain differential operator with respect to a variable x, the expansion of an arbitrary function in terms of the eigenfunctions of the differential equation

$$L\psi = \lambda\psi \tag{1.1}$$

could be obtained by considering the partial differential equation

$$L\psi = \frac{\partial\psi}{\partial t}. \tag{1.2}$$

The solution of this by means of Fourier integrals leads to the required expansion.

I now apply a similar method to some other cases.† The procedure adopted is, however, slightly different. It is seen that the success of the previous method depends on the fact that the 'spectrum' of the operator L is bounded in one direction. To include cases where this is not true we proceed as follows.

Consider the equation

$$L\psi = i\frac{\partial\psi}{\partial t} \quad (-\infty < t < \infty), \tag{1.3}$$

and suppose that this has a solution $\psi(x,t)$ which is $O(e^{A|t|})$ as $t \to \pm\infty$, and which reduces to a given function $\psi(x)$ when $t = 0$. Let

$$\Psi_+(x,w) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty \psi(x,t)e^{iwt}\,dt,$$

$$\Psi_-(x,w) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^0 \psi(x,t)e^{iwt}\,dt, \tag{1.4}$$

where w is a complex variable. These functions exist for $\mathbf{I}(w)$ sufficiently large and positive, and sufficiently large and negative, respectively. They are the functions which occur in the generalized

* *J. of London Math. Soc.* 14 (1939), 274–8.

† A general theory of these 'singular' cases is given by H. Weyl, *Math. Annalen*, 68 (1910), 220–69.

form of Fourier's integral theorem given in my book.* The reciprocal Fourier formula is

$$\psi(x,t) = \frac{1}{\sqrt{(2\pi)}} \int_{ia-\infty}^{ia+\infty} \Psi_+(x,w)e^{-iwt}\,dw + \frac{1}{\sqrt{(2\pi)}} \int_{ib-\infty}^{ib+\infty} \Psi_-(x,w)e^{-iwt}\,dw, \tag{1.5}$$

where a is a sufficiently large positive number, b a sufficiently large negative number.

We have formally

$$\begin{aligned} L\Psi_+ &= \frac{1}{\sqrt{(2\pi)}} \int_0^\infty L\psi e^{iwt}\,dt \\ &= \frac{i}{\sqrt{(2\pi)}} \int_0^\infty \frac{\partial \psi}{\partial t} e^{iwt}\,dt \\ &= \frac{i}{\sqrt{(2\pi)}} \left[\psi e^{iwt}\right]_0^\infty + \frac{w}{\sqrt{(2\pi)}} \int_0^\infty \psi e^{iwt}\,dt \\ &= -\frac{i}{\sqrt{(2\pi)}} \psi(x) + w\Psi_+, \end{aligned} \tag{1.6}$$

if $\psi(x,0) = \psi(x)$. This is an ordinary differential equation for Ψ_+, and conditions are imposed which determine it completely. Similarly, Ψ_- is determined, and $\psi(x,t)$ is then given by (1.5). The required expansion of $\psi(x)$ is obtained by using Cauchy's theorem in an appropriate way, and putting $t = 0$.

2. To illustrate the method in a familiar case, suppose that

$$L \equiv \frac{\partial^2}{\partial x^2} + \cot x \frac{\partial}{\partial x} \quad (0 < x < \pi). \tag{2.1}$$

A solution of

$$\frac{\partial^2 f}{\partial x^2} + \cot x \frac{\partial f}{\partial x} + \nu(\nu+1)f = 0 \tag{2.2}$$

is

$$f_\nu(x) = \frac{1}{2\pi} \int_C \frac{\cos(\nu+\frac{1}{2})\zeta}{(2\cos\zeta - 2\cos x)^{\frac{1}{2}}}\,d\zeta = \frac{1}{\pi} \int_{-x}^{x} \frac{\cos(\nu+\frac{1}{2})\zeta}{(2\cos\zeta - 2\cos x)^{\frac{1}{2}}}\,d\zeta, \tag{2.3}$$

where C is a closed contour surrounding $-x$ and x but excluding the other singularities of the integrand, and the value of the square root in the first formula is chosen so that, in the second formula,

* *Theory of Fourier Integrals*, §§ 1.3, 1.25.

the square root has its positive value. Here $f_\nu(x)$ is an even integral function of $\nu+\frac{1}{2}$, and so an integral function of w if $w = -\nu(\nu+1)$.

Another solution is plainly $g_\nu(x) = f_\nu(\pi-x)$. Also

$$f_\nu(\tfrac{1}{2}\pi) = \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos(\nu+\frac{1}{2})\zeta}{(2\cos\zeta)^{\frac{1}{2}}}\,d\zeta = \frac{\sqrt{\pi}}{\Gamma(1+\frac{1}{2}\nu)\Gamma(\frac{1}{2}-\frac{1}{2}\nu)},$$

and

$$f'_\nu(\tfrac{1}{2}\pi) = -\frac{1}{2\pi}\int_C \frac{\cos(\nu+\frac{1}{2})\zeta}{(2\cos\zeta)^{\frac{3}{2}}}\,d\zeta$$

$$= -\frac{1}{2\pi}\int_C \frac{\cos(\nu-\frac{1}{2})\zeta\cos\zeta - \sin(\nu-\frac{1}{2})\zeta\sin\zeta}{(2\cos\zeta)^{\frac{3}{2}}}\,d\zeta.$$

Integrating the second term by parts, we obtain

$$-\frac{\nu}{2\pi}\int_C \frac{\cos(\nu-\frac{1}{2})\zeta}{(2\cos\zeta)^{\frac{1}{2}}}\,d\zeta = \frac{-\nu\sqrt{\pi}}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu)\Gamma(1-\frac{1}{2}\nu)}.$$

Hence

$$f_\nu(\tfrac{1}{2}\pi)g'_\nu(\tfrac{1}{2}\pi) - g_\nu(\tfrac{1}{2}\pi)f'_\nu(\tfrac{1}{2}\pi) = -2f_\nu(\tfrac{1}{2}\pi)f'_\nu(\tfrac{1}{2}\pi) = \frac{2}{\pi}\sin\nu\pi,$$

and hence $$f_\nu(x)g'_\nu(x) - g_\nu(x)f'_\nu(x) = \frac{\phi(\nu)}{\sin x} = \frac{2\sin\nu\pi}{\pi\sin x}.$$

The general solution of (1.6) in this case is therefore

$$\Psi_+ = Af_\nu(x) + Bg_\nu(x) - \frac{i\sqrt{\pi}}{2^{\frac{1}{2}}\sin\nu\pi}\Big\{g_\nu(x)\int_0^x f_\nu(y)\sin y\,\psi(y)\,dy + \\ + f_\nu(x)\int_x^\pi g_\nu(y)\sin y\,\psi(y)\,dy\Big\}. \quad (2.4)$$

As $x \to 0$, $g_\nu(x) \to \infty$, so that we must take $B = 0$. Similarly, by considering $x \to \pi$, $A = 0$. Thus Ψ_+ is determined; and a similar argument shows that $\Psi_- = -\Psi_+$.

The function Ψ_+ has poles at the points $w = -n(n+1)$ ($n = 0, 1, \ldots$), corresponding to $\nu = n$. Now

$$f_n(x) = P_n(\cos x),$$

$$g_n(x) = P_n(-\cos x) = (-1)^n P_n(\cos x),$$

and

$$\sin\nu\pi = \pi\cos n\pi\frac{d\nu}{dw}\{w+n(n+1)\} + \ldots = \frac{(-1)^{n-1}\pi}{2n+1}\{w+n(n+1)\} + \ldots.$$

Hence (1.5) gives

$$\psi(x,t) = \sum_{n=0}^{\infty} e^{in(n+1)t}(n+\tfrac{1}{2})P_n(\cos x)\int_0^{\pi} P_n(\cos y)\sin y\,\psi(y)\,dy, \quad (2.5)$$

and the expansion of $\psi(x)$ follows if we may put $t = 0$, $\psi(x,0) = \psi(x)$.

3. The analysis so far must be regarded as purely formal, since we do not know *a priori* that (1.3) has a solution with the properties which have been assumed. We could use an argument similar to that of the previous paper to justify the process; but a more direct method is as follows.

To make the argument as simple as possible, suppose that $\psi(y)$ has continuous derivatives of the first and second order. Then by (2.2)

$$\int_0^x f_\nu(y)\sin y\,\psi(y)\,dy = -\frac{1}{\nu(\nu+1)}\int_0^x \left\{\frac{\partial^2 f_\nu}{\partial y^2}\sin y + \frac{\partial f_\nu}{\partial y}\cos y\right\}\psi(y)\,dy$$

$$= -\frac{1}{\nu(\nu+1)}\left\{\frac{\partial f_\nu}{\partial x}\sin x\,\psi(x) - f_\nu(x)\sin x\,\psi'(x) + \right.$$

$$\left. + \int_0^x f_\nu(y)\{\cos y\,\psi'(y) + \sin y\,\psi''(y)\}\,dy\right\}.$$

Similarly,

$$\int_x^{\pi} g_\nu(y)\sin y\,\psi(y)\,dy = -\frac{1}{\nu(\nu+1)}\left\{-\frac{\partial g_\nu}{\partial x}\sin x\,\psi(x) + g_\nu(x)\sin x\,\psi'(x) + \right.$$

$$\left. + \int_x^{\pi} g_\nu(y)\{\cos y\,\psi'(y) + \sin y\,\psi''(y)\}\,dy\right\}.$$

Hence, defining Ψ_+ by (2.4) with $A = B = 0$, we have

$$\Psi_+ = -\frac{i\psi(x)}{(2\pi)^{\frac{1}{2}}\nu(\nu+1)} +$$

$$+ \frac{i\sqrt{\pi}}{2^{\frac{1}{2}}\nu(\nu+1)\sin\nu\pi}\left\{g_\nu(x)\int_0^x f_\nu(y)\{\cos y\,\psi'(y) + \sin y\,\psi''(y)\}\,dy + \right.$$

$$\left. + f_\nu(x)\int_x^{\pi} g_\nu(y)\{\cos y\,\psi'(y) + \sin y\,\psi''(y)\}\,dy\right\}. \quad (3.1)$$

Now for large ν, $0 < x < \pi$,

$$f_\nu(x) = O\left\{\frac{e^{x|\mathrm{I}(\nu)|}}{|\nu\sin x|^{\frac{1}{2}}}\right\}. \quad (3.2)$$

For, if $|\mathbf{I}(\nu)| > A|\nu|$,

$$f_\nu(x) = \frac{1}{\pi}\int_0^x \frac{\cos(\nu+\frac{1}{2})\zeta}{\{\sin\frac{1}{2}(x+\zeta)\sin\frac{1}{2}(x-\zeta)\}^{\frac{1}{2}}}\,d\zeta = O\left\{\frac{1}{(\sin x)^{\frac{1}{2}}}\int_0^x \frac{e^{\zeta|\mathbf{I}(\nu)|}}{(x-\zeta)^{\frac{1}{2}}}\,d\zeta\right\}$$

$$= O\left\{\frac{e^{x|\mathbf{I}(\nu)|}}{(\sin x)^{\frac{1}{2}}}\int_0^\infty \frac{e^{-\zeta|\mathbf{I}(\nu)|}}{\xi^{\frac{1}{2}}}\,d\zeta\right\} = O\left\{\frac{e^{x|\mathbf{I}(\nu)|}}{|\mathbf{I}(\nu)\sin x|^{\frac{1}{2}}}\right\},$$

and (3.2) follows. Otherwise, i.e. for ν real or nearly so, we could write

$$f_\nu(x) = \frac{1}{\pi}\int_C \frac{e^{i(\nu+\frac{1}{2})\zeta}}{(2\cos\zeta - 2\cos x)^{\frac{1}{2}}}\,d\zeta = \frac{1}{\pi}\left(\int_{(-x+)}^{\pm i\infty} + \int_{(x+)}^{\pm i\infty}\right)$$

according as $\mathbf{R}(\nu)$ is positive or negative, and argue in a similar way with these integrals.

From (3.1) and (3.2) we obtain

$$\Psi_+ = -\frac{i\psi(x)}{(2\pi)^{\frac{1}{2}}\nu(\nu+1)} + O\left\{\frac{1}{|\nu|^3}\right\}$$

$$= \frac{i\psi(x)}{(2\pi)^{\frac{1}{2}}w} + O\left\{\frac{1}{|w|^{\frac{3}{2}}}\right\} \tag{3.3}$$

except in the neighbourhood of the poles. The usual integration round a large semicircle therefore gives

$$\lim_{R\to\infty}\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{i-R}^{i+R}\Psi_+(x,w)\,dw = \tfrac{1}{2}\psi(x).$$

Similarly for Ψ_-. The result

$$\psi(x) = \sum_{n=0}^{\infty}(n+\tfrac{1}{2})P_n(\cos x)\int_0^\pi P_n(\cos y)\sin y\,\psi(y)\,dy \tag{3.4}$$

therefore follows from the calculus of residues.

It is also easily seen that $\psi(x,t)$, defined by (1.5), satisfies (2.5), the series converging uniformly with respect to t; $\psi(x,t)$ is bounded as $t\to\pm\infty$ and satisfies (1.3), so that the conditions originally assumed actually hold.

The expansion (3.4) can be proved under the ordinary conditions for Fourier series by considering in more detail the nature of $f_\nu(x)$ for large ν. The general theorem on Sturm-Liouville series* can also be proved in a similar way.

* See E. L. Ince, *Ordinary Differential Equations*, 273–8.

4. The example

$$L \equiv \frac{\partial^2}{\partial x^2} + x \quad (0 < x < \infty)$$

is considered by Weyl* as a case where the spectrum covers the whole range $-\infty$ to ∞. Here solutions of $(L-w)f = 0$ are

$$f = (x-w)^{\frac{1}{2}} J_{\frac{1}{3}}(X), \qquad g = (x-w)^{\frac{1}{2}} Y_{\frac{1}{3}}(X),$$

writing for brevity

$$X = \tfrac{2}{3}(x-w)^{\frac{3}{2}}, \qquad Y = \tfrac{2}{3}(y-w)^{\frac{3}{2}}, \qquad W = \tfrac{2}{3}(-w)^{\frac{3}{2}}.$$

Also

$$f\frac{\partial g}{\partial x} - g\frac{\partial f}{\partial x} = \frac{3}{\pi}.$$

The general solution of (1.6) is therefore

$$\sqrt{(2\pi)}\Psi_+ = A(x-w)^{\frac{1}{2}} J_{\frac{1}{3}}(X) + B(x-w)^{\frac{1}{2}} H^{(2)}_{\frac{1}{3}}(X) +$$

$$+ \tfrac{1}{3}\pi(x-w)^{\frac{1}{2}} H^{(2)}_{\frac{1}{3}}(X) \int_0^x (y-w)^{\frac{1}{2}} J_{\frac{1}{3}}(Y)\psi(y)\, dy +$$

$$+ \tfrac{1}{3}\pi(x-w)^{\frac{1}{2}} J_{\frac{1}{3}}(X) \int_x^\infty (y-w)^{\frac{1}{2}} H^{(2)}_{\frac{1}{3}}(Y)\psi(y)\, dy.$$

This is the appropriate form since, for a fixed w in the upper half-plane,

$$(y-w)^{\frac{1}{2}} H^{(2)}_{\frac{1}{3}}(Y) \sim \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \frac{1}{(y-w)^{\frac{1}{4}}} \exp(-iY + \tfrac{5}{12}\pi i)$$

$$\sim \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \frac{1}{(y-w)^{\frac{1}{4}}} \exp\{-i(\tfrac{2}{3}y^{\frac{3}{2}} - wy^{\frac{1}{2}} - \tfrac{5}{12}\pi)\},$$

which is exponentially small as $y \to \infty$. All other solutions of the equation are exponentially large. Hence $A = 0$. The constant B is determined by the assumption that $\psi(0, t) = 0$ for all t, which gives $\Psi_+(0, w) = 0$ for all w. Hence

$$\sqrt{(2\pi)}\Psi_+ = -\tfrac{1}{3}\pi \frac{J_{\frac{1}{3}}(W)}{H^{(2)}_{\frac{1}{3}}(W)} (x-w)^{\frac{1}{2}} H^{(2)}_{\frac{1}{3}}(X) \int_0^\infty (y-w)^{\frac{1}{2}} H^{(2)}_{\frac{1}{3}}(Y)\psi(y)\, dy +$$

$$+ \tfrac{1}{3}\pi(x-w)^{\frac{1}{2}} H^{(2)}_{\frac{1}{3}}(X) \int_0^x (y-w)^{\frac{1}{2}} J_{\frac{1}{3}}(Y)\psi(y)\, dy +$$

$$+ \tfrac{1}{3}\pi(x-w)^{\frac{1}{2}} J_{\frac{1}{3}}(X) \int_x^\infty (y-w)^{\frac{1}{2}} H^{(2)}_{\frac{1}{3}}(Y)\psi(y)\, dy.$$

* Loc. cit. 267–8.

Similarly, Ψ_- is obtained by replacing $H_{\frac{1}{3}}^{(2)}$ by $H_{\frac{1}{3}}^{(1)}$. Substituting these formulae in (1.5), and making $a \to 0$, $b \to 0$, we obtain

$$\psi(x,t) = \frac{1}{3}\int_{-\infty}^{\infty} \frac{J_{\frac{1}{3}}(X)Y_{\frac{1}{3}}(W) - Y_{\frac{1}{3}}(X)J_{\frac{1}{3}}(W)}{H_{\frac{1}{3}}^{(1)}(W)H_{\frac{1}{3}}^{(2)}(W)} (x-w)^{\frac{1}{2}} e^{-itw}\, dw \times$$

$$\times \int_0^{\infty} \{J_{\frac{1}{3}}(Y)Y_{\frac{1}{3}}(W) - Y_{\frac{1}{3}}(Y)J_{\frac{1}{3}}(W)\}(y-w)^{\frac{1}{2}}\psi(y)\, dy.$$

Putting $t = 0$ we obtain Weyl's result.

A rigorous proof can be constructed as in the previous case by means of the known asymptotic expansions of the Bessel functions. We have, for example, for large w,

$$(x-w)^{\frac{1}{2}}H_{\frac{1}{3}}^{(2)}(X)(y-w)^{\frac{1}{2}}J_{\frac{1}{3}}(Y) \sim \frac{3e^{i(X-Y)}}{\pi(x-w)^{\frac{1}{2}}(y-w)^{\frac{1}{2}}} \sim \frac{3e^{i(-w)^{\frac{1}{2}}(y-x)}}{\pi(x-w)^{\frac{1}{2}}(y-w)^{\frac{1}{2}}},$$

where, to agree with the values already taken, $-\frac{1}{2}\pi < \arg(-w)^{\frac{1}{2}} < 0$ for $0 < \arg w < \pi$. The term involving this expression is therefore small in the upper half-plane.

5. Another interesting example is obtained by taking*

$$L \equiv \frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} + \frac{c}{x} - \frac{k(k+1)}{x^2} \quad (0 < x < \infty), \tag{5.1}$$

where $c > 0$, and k is a positive integer. A solution of $(L-w)f = 0$ is now

$$f(x,w) = \frac{1}{x^{k+1}}\int_{iK-\infty}^{iK+\infty} \frac{e^{-ix\zeta}}{(\zeta - i\sqrt{w})^{k+1-\alpha}(\zeta + i\sqrt{w})^{k+1+\alpha}}\, d\zeta, \tag{5.2}$$

where $\alpha = \frac{1}{2}c/\sqrt{w}$, $-\frac{1}{2}\pi \leqslant \arg\sqrt{w} \leqslant \frac{1}{2}\pi$, and $K > \mathbf{R}(\sqrt{w})$. Another solution, $g(x,w)$ say, is obtained by taking the integral between the singularities of the integrand, e.g. along the real axis if $\mathbf{R}(\sqrt{w}) > 0$. If $w = ue^{i\pi}$ $(u > 0)$ indentations are made above u and below $-u$, and *vice versa* if $w = ue^{-i\pi}$. In each case $(\zeta \pm i\sqrt{w})^{\alpha} = \exp\{\alpha \log(\zeta \pm i\sqrt{w})\}$, where the logarithm has its principal value when $\mathbf{R}(\zeta)$ is large and positive.

Since

$$\left(\frac{d}{d\zeta}\right)^{2k+1}(\zeta - i\sqrt{w})^{k+\alpha}(\zeta + i\sqrt{w})^{k-\alpha}$$

$$= \alpha(\alpha^2-1)\ldots(\alpha^2-k^2)(2i\sqrt{w})^{2k+1}(\zeta - i\sqrt{w})^{-k-1+\alpha}(\zeta + i\sqrt{w})^{-k-1-\alpha}$$

* See E. Schrödinger, *Abhandlungen zur Wellenmechanik* (Leipzig, 1927), 3–10; Hilbert-Courant, *Methoden der math. Physik*, I. 2. Aufl., 294–6.

we can integrate by parts $2k+1$ times, and obtain

$$f(x,w) = \frac{x^k}{c(c^2-4w)\ldots(c^2-4k^2w)} \int (\zeta - i\sqrt{w})^{k+\alpha}(\zeta+i\sqrt{w})^{k-\alpha}e^{-ix\zeta}\,d\zeta, \tag{5.3}$$

where the contour must now be bent into the lower half-plane or made into a closed curve; similarly for $g(x,w)$.

The differential equation now gives

$$f\frac{\partial g}{\partial x} - g\frac{\partial f}{\partial x} = \frac{\phi(w)}{x^2}, \tag{5.4}$$

where ϕ is a function of w only. Denoting the integrals in (5.2) and the corresponding formula for g by $p(x,w)$ and $q(x,w)$,

$$q\frac{\partial p}{\partial x} - p\frac{\partial q}{\partial x} = x^{2k}\phi(w). \tag{5.5}$$

Now $$\left\{\left(\frac{\partial}{\partial x}\right)^m p(x,w)\right\}_{x=0} = \int \frac{(-i\zeta)^m}{(\zeta-i\sqrt{w})^{k+1-\alpha}(\zeta+i\sqrt{w})^{k+1+\alpha}}\,d\zeta.$$

Taking the integral round a circle whose radius tends to infinity, we obtain the value 0 ($m \leqslant 2k$), $(-1)^k 2\pi$ ($m = 2k+1$). Hence, differentiating (5.5) $2k$ times and putting $x = 0$,

$$(-1)^k 2\pi q(0,w) = (2k)!\,\phi(w).$$

If $k < \mathbf{R}(\alpha) < k+1$,

$$q(0,w) = (1-e^{-2\pi i\alpha}) \int_0^\infty \frac{d\xi}{\xi^{k+1-\alpha}(\xi+2i\sqrt{w})^{k+1+\alpha}}$$

$$= \frac{1-e^{-2\pi i\alpha}}{(2i\sqrt{w})^{2k+1}} \int_0^\infty \frac{u^{2k}\,du}{(u+1)^{k+1+\alpha}} = \frac{-(2k)!\,(1-e^{-2\pi i\alpha})}{(\alpha-k)(\alpha-k+1)\ldots(\alpha+k)(2i\sqrt{w})^{2k+1}},$$

and the result follows by analytic continuation for all α. Hence

$$\phi(w) = \frac{-2\pi i(1-e^{-2\pi i\alpha})}{c(c^2-4w)\ldots(c^2-4k^2w)}. \tag{5.6}$$

The general solution of (1.6) is now

$$\Psi_+ = Af(x,w) + Bg(x,w) - \frac{ig(x,w)}{\sqrt{(2\pi)}\phi(w)} \int_0^x f(y,w)y^2\psi(y)\,dy -$$

$$- \frac{if(x,w)}{\sqrt{(2\pi)}\phi(w)} \int_x^\infty g(y,w)y^2\psi(y)\,dy.$$

Since $f(x,w)$ is large for large x, and $g(x,w)$ for small x, $A=0$ and $B=0$. Also $\Psi_- = -\Psi_+$.

The singularities of Ψ_+ are poles at

$$w = w_n = \frac{c^2}{4(n+k+1)^2} \quad (n = 0, 1, \ldots)$$

and a branch-point at $w=0$. It follows from Cauchy's theorem that $f(x,w_n) = g(x,w_n)$. Hence (1.5) gives formally

$$\psi(x,t) = -\sum_{n=0}^{\infty} \frac{e^{-iw_n t}}{\phi'(w_n)} f(x,w_n) \int_0^{\infty} f(y,w_n) y^2 \psi(y)\, dy +$$

$$+ \frac{1}{\sqrt{(2\pi)}} \int_0^{\infty} \{\Psi_+(x, ue^{i\pi}) + \Psi_-(x, ue^{-i\pi})\} e^{iut}\, du.$$

Now $$\phi'(w_n) = -\frac{8\pi^2}{c^{2k+3}} \frac{n!\,(n+k+1)^{2k+4}}{(n+2k+1)!}.$$

Also $f(x,w)$ is a one-valued function of w, while it is easily seen by considering the appropriate contours that, for real positive u,

$$g(x, ue^{i\pi}) = f(x,-u) - e^{-\pi c/\sqrt{u}} g(x, ue^{-i\pi}).$$

Hence

$$\frac{g(x, ue^{i\pi})}{\phi(ue^{i\pi})} - \frac{g(x, ue^{-i\pi})}{\phi(ue^{-i\pi})} = -\frac{c(c^2+4u)\ldots(c^2+4k^2u)}{2\pi i} \frac{f(x,-u)}{1-e^{-\pi c/\sqrt{u}}}.$$

Hence

$\psi(x,t)$

$$= \frac{c^{2k+3}}{8\pi^2} \sum_{n=0}^{\infty} \frac{(n+2k+1)!}{(n+k+1)^{2k+4} n!} e^{\frac{-it}{4(n+k+1)^2}} f(x,w_n) \int_0^{\infty} f(y,w_n) y^2 \psi(y)\, dy +$$

$$+ \frac{1}{4\pi^2} \int_0^{\infty} \frac{c(c^2+4u)\ldots(c^2+4k^2u)}{1-e^{-\pi c/\sqrt{u}}} e^{iut} f(x,-u)\, du \int_0^{\infty} f(y,-u) y^2 \psi(y)\, dy.$$

The expansion of $\psi(x)$ is obtained formally on putting $t=0$.

6. The function $f(x,w_n)$ can be evaluated in terms of Laguerre polynomials. We have

$$f(x,w_n) = \frac{1}{x^{k+1}} \int \left\{\zeta - \frac{i}{2(n+k+1)}\right\}^n \left\{\zeta + \frac{i}{2(n+k+1)}\right\}^{-n-2k-2} e^{-ix\zeta}\, d\zeta,$$

where the integral may now be taken round a closed contour including the pole. Putting

$$\zeta = -\frac{i}{2(n+k+1)} - \frac{i\zeta'}{x},$$

we obtain

$$f(x, w_n) = i(-x)^k e^{-\frac{x}{2(n+k+1)}} \int \left(\zeta' + \frac{x}{n+k+1}\right)^n \zeta'^{-n-2k-2} e^{-\zeta'}\, d\zeta'.$$

The Laguerre polynomial of degree p is

$$L_p(z) = e^z \left(\frac{d}{dz}\right)^p (z^p e^{-z}) = e^z \frac{p!}{2\pi i} \int \frac{\zeta^p e^{-\zeta}}{(\zeta - z)^{p+1}}\, d\zeta = \frac{p!}{2\pi i} \int \frac{(\zeta'+z)^p e^{-\zeta'}}{\zeta'^{p+1}}\, d\zeta'.$$

Hence
$$L_p^{(m)}(z) = \frac{(p!)^2}{2\pi i (p-m)!} \int \frac{(\zeta'+z)^{p-m}}{\zeta'^{p+1}} e^{-\zeta'}\, d\zeta'.$$

Hence

$$f(x, w_n) = \frac{(-1)^{k+1} n!\, 2\pi x^k e^{-\frac{x}{2(n+k+1)}}}{\{(n+2k+1)!\}^2} L_{n+2k+1}^{(2k+1)}\left(\frac{x}{n+k+1}\right). \quad (6.1)$$

The simplest formula for $f(x, -u)$ is obtained by taking the integral in (5.3) along the straight line joining $-i\sqrt{w}$ to $i\sqrt{w}$. We obtain

$$f(x, w) = \frac{(-x)^k w^{k+\frac{1}{2}} 4 \sin \frac{1}{2}\pi c/\sqrt{w}}{c(c^2-4w)\ldots(c^2-4k^2w)} \int_0^1 (1-r^2)^k \cosh\left(\frac{c}{2\sqrt{w}} \log\frac{1-r}{1+r} + xr\sqrt{w}\right) dr. \quad (6.2)$$

This gives a real formula for $f(x, -u)$.

If $c = 0$, $f(x, -u)$ reduces to

$$\frac{(-1)^k \pi^{\frac{1}{2}}}{k!\, 2^{k-\frac{1}{2}} x^{\frac{1}{2}} u^{\frac{1}{2}k+\frac{1}{4}}} J_{k+\frac{1}{2}}(x\sqrt{u}),$$

and the expansion reduces to Hankel's formula involving Bessel functions of order $k+\frac{1}{2}$.

7. To justify the process we have to consider Ψ_+ for large w and also for small w. For large w we proceed as in previous cases. By (6.2)

$$f(x, w) = O\left\{x^k \int_0^1 (1-r)^k e^{xr\mathbf{R}(\sqrt{w})}\, dr\right\}$$

$$= O\left\{x^k e^{x\mathbf{R}(\sqrt{w})} \int_0^1 \rho^k e^{-x\rho\mathbf{R}(\sqrt{w})}\, d\rho\right\} = O\left\{\frac{x^k e^{x\mathbf{R}(\sqrt{w})}}{\{x\mathbf{R}(\sqrt{w})\}^{k+1}}\right\}.$$

If $|\mathbf{R}(\sqrt{w})| > A|\sqrt{w}|$, it follows that

$$f(x,w) = O\left(\frac{e^{x|\mathbf{R}(\sqrt{w})|}}{x|w|^{\frac{1}{2}k+\frac{1}{2}}}\right).$$

As in § 3 this holds for the remaining values of w by deforming the r-integral. Similarly,

$$g(x,w) = O\left(\frac{e^{-x\mathbf{R}(\sqrt{w})}}{x|w|^{\frac{1}{2}k+\frac{1}{2}}}\right).$$

The procedure for large w is now valid if $\psi'(x)$ and $\psi''(x)$ are $L(0,\infty)$.

We have next to prove that

$$\lim_{n\to\infty} \int_{\gamma_n} \Psi_+(x,w)\,dw = 0$$

if γ_n is a suitable curve surrounding the origin and including the poles w_{n+1}, w_{n+2},... but excluding w_0,..., w_n. Suppose that $\psi(x) = 0$ for x sufficiently small or large, so that we need not consider $f(x,w)$ and $g(x,w)$ for $x \to 0$ or $x \to \infty$. Then $f(x,w)$ is bounded for small w; in fact, as $w \to 0$, $|\zeta| > A$,

$$\left(\frac{\zeta - i\sqrt{w}}{\zeta + i\sqrt{w}}\right)^{\alpha} \to e^{-ic/\zeta},$$

$$f(x,w) \to \frac{1}{x^{k+1}} \int_{iK-\infty}^{iK+\infty} \frac{e^{-ix\zeta - ic/\zeta}}{\zeta^{2k+2}}\,d\zeta = \frac{(-1)^n 2\pi J_{2n+1}(\sqrt{x})}{(2c)^{2n+1}\sqrt{x}}.$$

Also it is easily seen that

$$g(x,w) = h(x,w) - e^{-i\pi c/\sqrt{w}} f(x,w),$$

where

$$h(x,w) = \frac{1}{x^{k+1}} \int_{\infty}^{(i\sqrt{w}+)} \frac{e^{-ix\zeta}}{(\zeta - i\sqrt{w})^{k+1-\alpha}(\zeta + i\sqrt{w})^{k+1+\alpha}}\,d\zeta$$

$$= \frac{x^k}{c(c^2-4w)\ldots(c^2-4k^2w)} \int^{(i\sqrt{w}+)} (\zeta - i\sqrt{w})^{k+\alpha}(\zeta + i\sqrt{w})^{k-\alpha} e^{-ix\zeta}\,d\zeta,$$

where the line of integration is bent into the lower half-plane. Taking as part of the contour the straight line from $i\sqrt{w}$ to $i\sqrt{w}\,|w|^{-\frac{1}{2}}$ described twice, this part gives

$$(e^{-i\pi c/\sqrt{w}} - 1)(i\sqrt{w})^{2k+1} \int_1^{|w|^{-\frac{1}{2}}} (r-1)^{k+\alpha}(r+1)^{k-\alpha} e^{rx\sqrt{w}}\,dr$$

$$= O\left\{|e^{-i\pi c/\sqrt{w}} - 1|\,|w|^{k+\frac{1}{2}} \int_1^{|w|^{-\frac{1}{2}}} (r^2-1)^k\,dr\right\} = O(|e^{-i\pi c/\sqrt{w}} - 1|),$$

since $\mathbf{R}(\alpha) \geqslant 0$. The remainder of the integral can be taken so that $|\zeta| > A$, and then tends to a finite limit as in the case of $f(x, w)$. Hence for small w

$$g(x, w) = O(|e^{-i\pi c/\sqrt{w}}|) + O(1).$$

It is now easily seen that $\Psi_+(x, w)$ is bounded on a system of curves such as is required, and the expansion of $\psi(x)$ follows.

The conditions imposed on $\psi(x)$ can no doubt be relaxed, but as this seems to require a good deal more discussion it will not be attempted here.

8. It may be worth while noticing how some comparatively trivial cases are included in the method. Let

$$L \equiv i\frac{\partial}{\partial x} \quad (-\infty < x < \infty).$$

Then (1.6) gives

$$\Psi_+(x, w) = -\frac{e^{-iwx}}{\sqrt{(2\pi)}} \int_0^x \psi(y) e^{iwy}\, dy + Ce^{-iwx},$$

where C is independent of x. Since this must be bounded for a fixed w, $\mathbf{I}(w) > 0$, and $x \to \infty$,

$$C = \frac{1}{\sqrt{(2\pi)}} \int_0^\infty \psi(y) e^{iwy}\, dy.$$

Hence

$$\Psi_+ = \frac{e^{-iwx}}{\sqrt{(2\pi)}} \int_x^\infty \psi(y) e^{iwy}\, dy.$$

Similarly,

$$\Psi_- = \frac{e^{-iwx}}{\sqrt{(2\pi)}} \int_{-\infty}^x \psi(y) e^{iwy}\, dy.$$

Substituting in (1.5) and making $a \to 0$, $b \to 0$,

$$\psi(x, t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i(x+t)w}\, dw \int_{-\infty}^\infty \psi(y) e^{iwy}\, dy,$$

i.e. the expansion is Fourier's integral formula.

If $$L \equiv x$$

then $$\Psi_+ = \frac{i\psi(x)}{\sqrt{(2\pi)(w-x)}}, \qquad \Psi_- = \frac{-i\psi(x)}{\sqrt{(2\pi)(w-x)}},$$

and (1.5) gives $$\psi(x, t) = \psi(x) e^{-itx}.$$

The 'expansion' is therefore the identity $\psi(x) = \psi(x)$.

ON EXPANSIONS IN EIGENFUNCTIONS (III)

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1. In this note I consider the expansion of an arbitrary function in terms of Hermite polynomials.

We proceed formally as in the previous paper. Consider the solution $\psi(x,t)$ of

$$L\psi \equiv \frac{\partial^2\psi}{\partial x^2} - x^2\psi = i\frac{\partial\psi}{\partial t} \tag{1.1}$$

which vanishes as $x \to \pm\infty$, and reduces to a given function $\psi(x)$ when $t = 0$. If

$$\Psi_+(x,w) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty \psi(x,t)e^{iwt}\,dt, \tag{1.2}$$

we obtain as before

$$L\Psi_+ = -i\psi(x)/\sqrt{(2\pi)} + w\Psi_+. \tag{1.3}$$

Now two (in general) independent solutions of $(L-w)f = 0$ are

$$f(x,w) = \Gamma(\tfrac{1}{2}-\tfrac{1}{2}w)e^{-\frac{1}{4}x^2}\int_\infty^{(0+)} e^{-x\zeta-\frac{1}{2}\zeta^2}\zeta^{\frac{1}{2}w-\frac{1}{2}}\,d\zeta \tag{1.4}$$

and $g(x,w) = f(-x,w)$. Here

$$\zeta^{\frac{1}{2}w-\frac{1}{2}} = \exp\{(\tfrac{1}{2}w-\tfrac{1}{2})\log\zeta\},$$

where the logarithm starts at infinity with its imaginary part zero. Also

$$f(0,w) = \Gamma(\tfrac{1}{2}-\tfrac{1}{2}w)\int_\infty^{(0+)} e^{-\frac{1}{2}\zeta^2}\zeta^{\frac{1}{2}w-\frac{1}{2}}\,d\zeta$$

$$= -\Gamma(\tfrac{1}{2}-\tfrac{1}{2}w)(e^{i\pi w}+1)2^{\frac{1}{2}w-\frac{1}{2}}\Gamma(\tfrac{1}{4}+\tfrac{1}{4}w),$$

and similarly

$$f_x(0,w) = \Gamma(\tfrac{1}{2}-\tfrac{1}{2}w)(e^{i\pi w}+1)2^{\frac{1}{2}w+\frac{1}{2}}\Gamma(\tfrac{3}{4}+\tfrac{1}{4}w).$$

From the differential equation, $fg_x - gf_x$ is independent of x, and is therefore equal to

$$f(0,w)g_x(0,w) - g(0,w)f_x(0,w) = -2f(0,w)f_x(0,w)$$

$$= 2^{\frac{1}{2}+\frac{1}{2}w}\pi^{\frac{1}{2}}e^{i\pi w}/\Gamma(\tfrac{1}{2}+\tfrac{1}{2}w) = \phi(w),$$

say, by the usual Γ-function formulae. The general solution of (1.3) is therefore

$$\Psi_+ = Af(x,w)+Bg(x,w)+ \\ +\frac{i}{\sqrt{(2\pi)\phi(w)}}\left\{f(x,w)\int_{-\infty}^{x} g(y,w)\psi(y)\,dy + g(x,w)\int_{x}^{\infty} f(y,w)\psi(y)\,dy\right\}, \tag{1.5}$$

where A and B are independent of x. To make Ψ_+ bounded as $x \to -\infty$ we must take $A = 0$ (the part of the integral (1.4) in the neighbourhood of $\zeta = -2x$ gives a term $e^{\frac{1}{2}x^2}$). Similarly, to make Ψ_+ bounded as $x \to \infty$ we must take $B = 0$.

Also clearly

$$\Psi_-(x,w) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{0} \psi(x,t)e^{iwt}\,dt = -\Psi_+(x,w).$$

Hence

$$\psi(x,t) = \frac{1}{\sqrt{(2\pi)}}\left(\int_{ia-\infty}^{ia+\infty} - \int_{ib-\infty}^{ib+\infty}\right)\Psi_+(x,w)e^{-itw}\,dw \quad (a>0,\ b<0). \tag{1.6}$$

Now Ψ_+ has poles at $w = -2n-1$ ($n = 0, 1, \ldots$). At these points $1/\phi(w)$ has the residues

$$(-1)^{n-1}2^{n-2}\pi^{-\frac{1}{2}}/n!.$$

Also

$$f(x,-2n-1) = n!\,e^{-\frac{1}{2}x^2}\int_{(0+)} e^{-x\zeta-\frac{1}{2}\zeta^2}\zeta^{-n-1}\,d\zeta = \frac{n!\,e^{\frac{1}{2}x^2}}{2^n}\int_{(x+)}\frac{e^{-\zeta'^2}}{(\zeta'-x)^{n+1}}\,d\zeta' \\ = 2\pi i\frac{e^{\frac{1}{2}x^2}}{2^n}\left(\frac{d}{dx}\right)^n e^{-x^2} = \frac{2\pi i}{2^n}(-1)^n e^{-\frac{1}{2}x^2}H_n(x),$$

where $H_n(x)$ is the Hermite polynomial of degree n; and

$$g(x,-2n-1) = (-1)^n f(x,-2n-1).$$

The calculus of residues applied to (1.6) therefore gives

$$\psi(x,t) = \sum_{n=0}^{\infty}\frac{e^{(2n+1)it}}{n!\,2^n\pi^{\frac{1}{2}}}e^{-\frac{1}{2}x^2}H_n(x)\int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2}H_n(y)\psi(y)\,dy. \tag{1.7}$$

The expansion of $\psi(x)$ follows on putting $t = 0$.

2. The above result will now be justified for a certain class of functions $\psi(x)$.

If $\psi(x)$ belongs to L^2 over a finite interval $(-X, X)$, and is zero

outside this interval, Ψ_+ and any of its derivatives are exponentially small as $x \to \pm\infty$. Hence

$$\int_{-\infty}^{\infty} \bar{\Psi}_+ . L\Psi_+ \, dx = \int_{-\infty}^{\infty} \Psi_+ . L\bar{\Psi}_+ \, dx$$

by integration by parts. Using (1.3) this gives

$$(w-\bar{w}) \int_{-\infty}^{\infty} |\Psi_+|^2 \, dx = \frac{i}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} (\Psi_+ + \bar{\Psi}_+)\psi(x) \, dx.$$

Hence, if $w = u+iv$, $v > 0$,

$$2v \int_{-\infty}^{\infty} |\Psi_+|^2 \, dx \leqslant \frac{2}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} |\Psi_+ \psi| \, dx$$

$$\leqslant \frac{2}{\sqrt{(2\pi)}} \left\{ \int_{-\infty}^{\infty} |\Psi_+|^2 \, dx \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx \right\}^{\frac{1}{2}},$$

so that* $$\int_{-\infty}^{\infty} |\Psi_+|^2 \, dx \leqslant \frac{1}{2\pi v^2} \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx. \tag{2.1}$$

This also holds for any ψ of $L^2(-\infty, \infty)$, since, distinguishing the above functions by a suffix X,

$$\lim_{X\to\infty} \int_{-c}^{c} |\Psi_{+,X}|^2 \, dx = \int_{-c}^{c} |\Psi_+|^2 \, dx$$

for a fixed c. Now

$$\int_{-c}^{c} |\Psi_{+,X}|^2 \, dx \leqslant \int_{-\infty}^{\infty} |\Psi_{+,X}|^2 \, dx \leqslant \frac{1}{2\pi v^2} \int_{-\infty}^{\infty} |\psi_X|^2 \, dx \leqslant \frac{1}{2\pi v^2} \int_{-\infty}^{\infty} |\psi|^2 \, dx.$$

The general result therefore follows on making first $X \to \infty$, then $c \to \infty$.

Again, if $\xi < x$,

$$(x-\xi)^2\Psi_+(x) = \int_{\xi}^{x} (6y-2x-4\xi)\Psi_+(y) \, dy + \int_{\xi}^{x} (x-y)(y-\xi)^2\Psi''_+(y) \, dy$$

by integration by parts; and by (1.3) the last term is equal to

$$\int_{\xi}^{x} (x-y)(y-\xi)^2 \left\{ y^2\Psi_+(y) + w\Psi_+(y) - \frac{i\psi(y)}{\sqrt{(2\pi)}} \right\} dy.$$

* The proof of (2.1) is taken from M. H. Stone, *Linear Transformations in Hilbert Space*, 143.

Now

$$\left|\int_\xi^x (x-y)(y-\xi)^2\Psi_+(y)\,dy\right| \leqslant (x-\xi)^3\left\{\int_\xi^x dy\int_\xi^x |\Psi_+(y)|^2\,dy\right\}^{\frac12}$$

$$\leqslant (x-\xi)^{\frac52}\left\{\int_{-\infty}^{\infty}|\Psi_+(y)|^2\,dy\right\}^{\frac12} \leqslant C(x-\xi)^{\frac52}v^{-1},$$

where

$$C=\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}|\psi(x)|^2\,dx\right\}^{\frac12}.$$

Similarly,

$$\left|\int_\xi^x (x-y)(y-\xi)^2y^2\Psi_+(y)\,dy\right| \leqslant C(x-\xi)^{\frac52}\max(x^2,\xi^2)v^{-1},$$

$$\left|\int_\xi^x (x-y)(y-\xi)^2\frac{\psi(y)}{\sqrt{(2\pi)}}\,dy\right| \leqslant C(x-\xi)^{\frac52},$$

$$\left|\int_\xi^x (6y-2x-4\xi)\Psi_+(y)\,dy\right| \leqslant 4C(x-\xi)^{\frac32}v^{-1}.$$

Hence

$$|\Psi_+(x)| \leqslant C(x-\xi)^{\frac52}\left\{\frac{|w|}{v}+\frac{\max(x^2,\xi^2)}{v}+1\right\}+\frac{4C}{(x-\xi)^{\frac12}v}.$$

To make the first and last terms of the same order, take $\xi = x-|w|^{-\frac12}$. We obtain, for a fixed x, $|w|>A$, $v>0$,

$$\Psi_+(x) = O(|w|^{\frac12}v^{-1}). \tag{2.2}$$

3. Now suppose that ψ and ψ' are absolutely continuous, and that $\psi''(x)-x^2\psi(x)$ is $L^2(-\infty,\infty)$. Then

$$\int_x^\infty f(y,w)\psi(y)\,dy = \frac1w\int_x^\infty\left(\frac{\partial^2 f}{\partial y^2}-y^2f\right)\psi(y)\,dy$$

$$=\frac1w\left\{-\frac{\partial f}{\partial x}\psi(x)+f(x)\psi'(x)+\int_x^\infty f(y)\{\psi''(y)-y^2\psi(y)\}\,dy\right\},$$

and similarly for the other integral in (1.5). Hence

$$\Psi_+ = \frac{i\psi(x)}{\sqrt{(2\pi)}w}+\frac{i}{\sqrt{(2\pi)}w\,\phi(w)}\left\{f(x,w)\int_{-\infty}^x g(y,w)\{\psi''(y)-y^2\psi(y)\}\,dy+\right.$$

$$\left.+g(x,w)\int_x^\infty f(y,w)\{\psi''(y)-y^2\psi(y)\}\,dy\right\}$$

$$=\frac{i\psi(x)}{\sqrt{(2\pi)}w}+O\left(\frac{1}{v|w|^{\frac12}}\right), \tag{3.1}$$

by (2.2) with $\psi''-y^2\psi$ instead of ψ. The usual integration round a large semicircle therefore gives

$$\frac{1}{\sqrt{(2\pi)}}\int_{ia-\infty}^{ia+\infty} \Psi_+(x,w)\,dw = \tfrac{1}{2}\psi(x) \quad (a>0).$$

Similarly,

$$-\frac{1}{\sqrt{(2\pi)}}\int_{ib-\infty}^{ib+\infty} \Psi_+(x,w)\,dw = \tfrac{1}{2}\psi(x) \quad (b<0).$$

Again, *if* $F(w)$ *is regular and* $|F(w)| \leqslant M/|v|$ *for* $-c \leqslant u \leqslant c$, $-c \leqslant v \leqslant c$, *then* $|F(w)| \leqslant 3M/c$ *for* $u = 0$, $-c \leqslant v \leqslant c$. This is easily seen by applying the principle of the maximum modulus to $(w^2-c^2)F(w)$. We deduce that

$$\Psi_+(x,w) = O(|w|^{-\frac{1}{2}})$$

on the segments of $u = \pm 2, \pm 4, \ldots$ in the strip. This justifies the application of the calculus of residues to obtain the series (in the case $t = 0$). Hence under the conditions stated

$$\psi(x) = \sum_{n=0}^{\infty} \frac{e^{-\frac{1}{2}x^2}H_n(x)}{n!\,2^n\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2}H_n(y)\psi(y)\,dy. \tag{3.2}$$

Proofs of this formula under more general conditions have been given by Weyl* and Szegö.† No doubt it would be possible to extend the present method by a more detailed discussion of the functions involved. The main point of the proof given is that it can easily be extended to other expansions.

4. If we merely require a convergence-in-mean result, we can argue similarly with the function

$$\int_{-\infty}^{\infty} \psi(x)\Psi_+(x,w)\,dx$$

instead of Ψ_+. We obtain the Parseval formula

$$\int_{-\infty}^{\infty} \{\psi(x)\}^2\,dx = \sum_{n=0}^{\infty} \frac{1}{n!\,2^n\pi^{\frac{1}{2}}} \left\{ \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}H_n(x)\psi(x)\,dx \right\}^2.$$

The process is justified if $\psi''-x^2\psi$ is L^2 by (2.1), it being unnecessary to prove (2.2). The mean-square theory for all ψ of L^2 can then be deduced by familiar arguments.

* H. Weyl, *Math. Annalen*, 66 (1909), 273–324 (306).

† G. Szegö, *Math. Zeitschrift*, 25 (1926), 87–115.

ON THE FRENET FORMULAE FOR A V_m IN A V_n

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1. Introduction

In a V_n* the Frenet formulae† for a curve (C) $\xi^\kappa = \xi^\kappa(s)$ (s = arc length)

$$\underset{x}{\delta_s} \underset{x}{i^\kappa} = \underset{x}{\epsilon}\Big(-\underset{x}{\kappa}\ \underset{x-1}{i^\kappa} + \underset{x}{\kappa}\ \underset{x+1}{i^\kappa}\Big),$$

$$\delta_s = i^\mu \nabla_\mu, \qquad \underset{0}{i^\kappa} = i^\kappa = \frac{d\xi^\kappa}{ds}, \qquad \underset{x}{\epsilon} = \underset{x}{i^\kappa}\,\underset{x}{i_\kappa} \equiv a_{\kappa\lambda}\,\underset{x}{i^\kappa}\,\underset{x}{i^\lambda} = \pm 1$$

(x not summed),

$$\underset{0}{\kappa} = \underset{k+1}{\kappa} = \ldots = \underset{n}{\kappa} = 0 \quad (x = 0, 1, \ldots, k;\ k+1 \leqslant n) \qquad (1.1)$$

are usually established without being given any geometrical meaning. But as early as 1922 Struik,‡ in treating the case when V_n is ordinary, hinted at this point. He called

$$\Big(\underset{0}{\delta_s i^\kappa}\Big)_m = \Big\{\Big(\underset{0}{\delta_s i^\kappa}\Big)\Big(\underset{0}{\delta_s i_\kappa}\Big)\Big\}^{\frac{1}{2}} = \frac{d\phi_1}{ds},$$

$$\Big(\underset{0}{\delta_s}\,\underset{1}{i^{[\kappa_1}}\,\underset{1}{i^{\kappa_2]}}\Big)_m = \Big\{2!\Big(\underset{0}{\delta_s i^{[\kappa_1}}\,\underset{1}{i^{\kappa_2]}}\Big)\Big(\underset{0}{\delta_s}\,\underset{1}{i_{[\kappa_1}}\,\underset{1}{i_{\kappa_2]}}\Big)\Big\}^{\frac{1}{2}} = \frac{d\phi_2}{ds}, \quad \text{etc.},$$

the 'geodätische Richtungsänderung' with respect to s of the tangent, the first osculating space, etc., and obtained

$$\frac{d\phi_x}{ds} = \underset{x}{\kappa}, \qquad (1.2)$$

although he did not give any concrete geometrical explanation as to what 'geodätische Richtungsänderung' really meant. Writing $d\epsilon_x/ds = (\underset{x}{\delta_s i^\kappa})_m$, he also gave

$$\left(\frac{d\underset{x}{\epsilon}}{ds}\right)^2 = \underset{x}{\kappa^2} + \underset{x+1}{\kappa^2} \qquad (1.3)$$

* We denote by V_n a real Riemannian n-space with coordinates ξ^κ ($\kappa = 1, \ldots, n$), fundamental tensor $a_{\kappa\lambda}$, and symbol of covariant differentiation ∇_μ. A V_n is said to be ordinary if $a_{\kappa\lambda}$ is positive definite.

† Here $\underset{1}{\kappa}, \ldots, \underset{k}{\kappa}$; $\underset{1}{i^\kappa}, \ldots, \underset{k}{i^\kappa}$ are respectively the curvatures and the unit principal normal vectors of (C) relative to V_n. See, for example, Schouten-Struik (**8**), II, 14, (2.15); Eisenhart (**4**), 106, (32.16).

‡ Struik (**9**), 72–7.

and the generalized Lancret's formula

$$\left(\frac{d\epsilon_0}{ds}\right)^2-\left(\frac{d\epsilon_1}{ds}\right)^2+\ldots=0 \quad (n>2). \tag{1.4}$$

A little later in 1924 P. Dienes,* by means of his theory of transported affinors and the definition of the angle between two affinors, succeeded in proving that the $(x+1)$th curvature at a point $P(s_0)$ of a curve (C) in a V_n, which he defines as the arc-rate change of the angle between the xth osculating space $R_{x+1}(s_0)$ and the linear space $R_{x+1}(C;s\,\|\,s_0)$† as s approaches s_0, is identical with the $\underset{x+1}{\kappa}$ in the Frenet formulae. About 1930 Hlavatý‡ arrived at more or less the same idea; he used the angle between $i^{\kappa}(s)$ and the auto-parallel vector $i^{\kappa}_{x-1}(C;s_0\,\|\,s)$ instead. In 1934 H. Levy,§ independently of the others, gave a similar geometrical interpretation to formula (1.3).

The purpose of this note is to generalize (1.3), which is concerned with a single principal normal or with the tangent of a curve, to the case of a V_m in a V_n. The resulting formula gives a geometrical interpretation to the symmetric affinors $\overset{x+1}{H}_{r_0\lambda}{}^{\kappa}\overset{x+1}{H}_{p_0\cdot\kappa}{}^{\lambda}$ of V_m. It contains moreover, for the case of a curve, both (1.2) and (1.3) as particular cases. The method which I use here is a generalization of Hlavatý's. Throughout this note when we consider any function of ξ^{κ} or of s, it is understood that it is real and continuous, as well as its derivatives of such order as appear in the discussion.

2. R_m and auto-parallel R_m along a curve**

In this section I give some preliminaries which are necessary for the later work. Although most of the results which we shall obtain

* Dienes (**2**), § 21, where he considered the more general case of the curvatures of a vector along a curve in a 'linearly connected tensor space'.

† In general, we denote by $Q(C;s\,\|\,s_0)$ the geometric object obtained by transporting a geometric object $Q(s)$ parallelly from $P(s)$ to $P(s_0)$ along (C); evidently $Q(C;s_0\,\|\,s_0)=Q(s_0)$.

‡ Hlavatý (**5**), 7–9, where he considered the case when V_n is ordinary.

§ Levy (**6**).

** We denote by R_m $(m<n)$ a linear vector m-space in the local (linear vector space) R_n of V_n not tangent to the null-cone of R_n. Thus in an R_m there exist infinitely many (real) m-Beins; and conversely, an m-Bein spans an R_m. Here by an m-Bein we mean a set of m mutually orthogonal unit vectors.

are more or less known, they will appear here in a form more suitable for our purpose.

Let $\underset{a}{j^\kappa}$ $(a, b, c, d = 1,..., m)$ with

$$\underset{a}{j^\kappa}\,\underset{\kappa}{j^b} \equiv a_{\kappa\lambda}\underset{a}{j^\kappa}\underset{b}{j^\lambda} = \underset{ab}{\epsilon} = \begin{cases} 0 & (a \neq b), \\ \underset{a}{\epsilon} = \pm 1 & (a = b), \end{cases} \tag{2.1}$$

$$\epsilon = \underset{1}{\epsilon}...\underset{m}{\epsilon}$$

be an m-Bein in an R_m, and let the (simple) m-vector $j^{\kappa_1\cdots\kappa_m}$ be formed from $\underset{a}{j^\kappa}$ by the equation

$$j^{\kappa_1\cdots\kappa_m} = \underset{1}{j^{[\kappa_1}}...\underset{m}{j^{\kappa_m]}} = \frac{1}{m!}e^{a_1\cdots a_m}\underset{a_1}{j^{\kappa_1}}...\underset{a_m}{j^{\kappa_m}} = \frac{1}{m!}\begin{vmatrix} \underset{1}{j^{\kappa_1}} & . & . & . & \underset{1}{j^{\kappa_m}} \\ . & . & . & . & . \\ \underset{m}{j^{\kappa_1}} & . & . & . & \underset{m}{j^{\kappa_m}} \end{vmatrix}. \tag{2.2}$$

We shall now consider the effect on $j^{\kappa_1\cdots\kappa_m}$ of a transformation from $\underset{a}{j^\kappa}$ to another m-Bein $\underset{a}{'j^\kappa}$ in R_m, for which $\underset{ab}{'\epsilon}$, $\underset{a}{'\epsilon}$, and $'\epsilon$ are defined by equations analogous to (2.1). Let the transformation be

$$T: \qquad \underset{a}{'j^\kappa} = \underset{ac}{c}\,\underset{a}{j^\kappa},$$

where it is seen that the m^2 scalars $\underset{a}{c}$ are uniquely determined as soon as $\underset{a}{j^\kappa}$ and $\underset{a}{'j^\kappa}$ are given. Then we have

$$'j^{\kappa_1\cdots\kappa_m} = \Delta\,.j^{\kappa_1\cdots\kappa_m}, \qquad \Delta = \det\left(\underset{a}{c}\right). \tag{2.3}$$

From $$a_{\kappa\lambda}\,\underset{a}{'j^\kappa}\,\underset{b}{'j^\lambda} = a_{\kappa\lambda}\underset{ac}{c}\,\underset{a}{j^\kappa}\,\underset{bd}{c}\,\underset{b}{j^\lambda}$$

and (2.1), it follows that

$$\underset{ab}{'\epsilon} = \epsilon\,\underset{cd}{c}\,\underset{a}{c}\,\underset{b}{c},$$

and thus $$\det\left(\underset{ab}{'\epsilon}\right) = \det\left(\underset{ab}{\epsilon}\right)\left\{\det\left(\underset{d}{c}\right)\right\}^2,$$

that is $$'\epsilon = \epsilon\Delta^2.$$

Since only real quantities are considered, the last equation is equivalent to

$$'\epsilon = \epsilon, \tag{2.4}$$

$$\Delta = \pm 1. \tag{2.5}$$

Hence, combining (2.3) with (2.5), we have

THEOREM 2.1. *Given an R_m the m-vector $j^{\kappa_1\cdots\kappa_m}$ is determined to within a sign.*

A converse of this theorem is

THEOREM 2.2. *If the m-vectors $j^{\kappa_1\cdots\kappa_m}$ and $'j^{\kappa_1\cdots\kappa_m}$ formed from the two m-Beins $\underset{a}{j^\kappa}$ and $\underset{a}{'j^\kappa}$ differ from each other by a scalar factor, then $\underset{a}{j^\kappa}$ and $\underset{a}{'j^\kappa}$ span the same R_m.*

The proof is easy. From

$$'j^{\kappa_1\cdots\kappa_m} = \rho j^{\kappa_1\cdots\kappa_m}$$

we have $$'j^{\kappa_1\cdots\kappa_m}\overset{2}{'j}_{\kappa_2}\cdots\overset{m}{'j}_{\kappa_m} = \rho j^{\kappa_1\cdots\kappa_m}\overset{2}{'j}_{\kappa_2}\cdots\overset{m}{'j}_{\kappa_m}.$$

Therefore $\underset{1}{'j^\kappa}$ is a linear combination of $\underset{a}{j^\kappa}$, and in like manner the same is true of each of $\underset{a}{'j^\kappa}$.

Equation (2.4), which will be used later, shows that ϵ is independent of the choice of the m-Bein $\underset{a}{j^\kappa}$ in R_m.* For this reason we call ϵ the *signum* of R_m.

Let (C) be a curve in a V_n with arc-length s and unit tangent vector i^κ. An $R_m(s)$ defined along (C) is said to be *auto-parallel*† [along (C)], if it contains m independent auto-parallel vectors.

Let $\underset{a}{u^\kappa}(s)$ $(a, b, c, d = 1,\ldots, m)$ be m independent auto-parallel vectors contained in an auto-parallel $R_m(s)$, then

$$\delta_s \underset{a}{u^\kappa}(s) = 0, \qquad \delta_s = i^\mu\nabla_\mu. \tag{2.6}$$

If $u^\kappa(s_0)$ is a vector in $R_m(s_0)$, there exist m constants $\underset{c}{a}$ such that $u^\kappa(s_0) = \underset{c}{a}\underset{c}{u^\kappa}(s_0)$. Now the vector $\underset{c}{a}\underset{c}{u^\kappa}(s)$, which lies in $R_m(s)$, coincides with $u^\kappa(s_0)$ at $P(s_0)$, and, because

$$\delta_s\Big\{\underset{c}{a}\underset{c}{u^\kappa}(s)\Big\} = \underset{c}{a}\delta_s \underset{c}{u^\kappa}(s) = 0$$

* Since the transformation T corresponds to a real non-singular linear transformation from one normal form of a quadratic form to another, this confirms the law of inertia which states that the numbers of positive and negative $\underset{a}{\epsilon}$ are invariant under such a transformation.

† Cf. Duschek-Mayer (3), 122–3, where, for the case of an ordinary V_n, a condition for the auto-parallelism of an $R_m(s)$ with fundamental tensor $'a_{\kappa\lambda}$ is given as $\delta_s\, 'a_{\kappa\lambda} = 0$.

by (2.6), it is auto-parallel along (C). Therefore $\overset{c}{a}\overset{c}{u}^{\kappa}(s) = u^{\kappa}(C; s_0 \| s)$ and we have

THEOREM 2.3. *An auto-parallel $R_m(s)$ has the property that, if, at any point $P(s_0)$, $u^{\kappa}(s_0)$ is an arbitrary vector in $R_m(s_0)$, then the auto-parallel vector $u(C; s_0 \| s)$ lies in $R_m(s)$ all along (C).*

We shall now prove

THEOREM 2.4. *Let $j^{\kappa_1 \ldots \kappa_m}(s)$ be formed from an m-Bein $\underset{a}{j}{}^{\kappa}(s)$ in an $R_m(s)$, then $R_m(s)$ is auto-parallel if and only if*

$$\delta_s j^{\kappa_1 \ldots \kappa_m}(s) = 0.^* \tag{2.7}$$

Proof. Since parallelism preserves length and orthogonality, it is always possible (and in an infinite number of ways), by Theorem 2.3, to find in an auto-parallel $R_m(s)$ an m-Bein $\underset{a}{'j}{}^{\kappa}(s)$ such that each of the vectors $\underset{a}{'j}{}^{\kappa}(s)$ is auto-parallel, i.e. $\delta_s \underset{a}{'j}{}^{\kappa}(s) = 0$. Hence, by Theorem 2.1,

$$\delta_s j^{\kappa_1 \ldots \kappa_m}(s) = \pm \delta_s {}'j^{\kappa_1 \ldots \kappa_m}(s) = \pm \sum_a {}'j^{[\kappa_1} \ldots \left(\delta_s \underset{a}{'j}{}^{\kappa_a}\right) \ldots {}'j^{\kappa_m]} = 0.$$

Conversely, let $j^{\kappa_1 \ldots \kappa_m}(s)$ formed from an m-Bein $\underset{a}{j}{}^{\kappa}(s)$ satisfy (2.7). Then the m-vector $j^{\kappa_1 \ldots \kappa_m}(C; s_0 \| s)$ formed from the m-Bein $\underset{a}{j}{}^{\kappa}(C; s_0 \| s)$, which spans an auto-parallel R_m, also satisfies (2.7). Now both $j^{\kappa_1 \ldots \kappa_m}(s)$ and $j^{\kappa_1 \ldots \kappa_m}(C; s_0 \| s)$ are solutions of the system of ordinary differential equations (2.7), and, since $\underset{a}{j}{}^{\kappa}(C; s_0 \| s_0) = \underset{a}{j}{}^{\kappa}(s_0)$ and consequently

$$j^{\kappa_1 \ldots \kappa_m}(C; s_0 \| s_0) = j^{\kappa_1 \ldots \kappa_m}(s_0), \tag{2.8}$$

they have the same value at $P(s_0)$. Therefore, they must be identical. Hence the $R_m(s)$ spanned by $\underset{a}{j}{}^{\kappa}(s)$, being identical with that spanned by $\underset{a}{j}{}^{\kappa}(C; s_0 \| s)$ by Theorem 2.2, is auto-parallel.

3. Angle between $R_m(s)$ and $R_m(C; s_0 \| s)$; a generalization of a formula of Hlavatý's

Let the auto-parallel $R_m(C; s_0 \| s)$ be obtained from an $R_m(s)$, then, if $\underset{a}{j}{}^{\kappa}(s)$ is an m-Bein in $R_m(s)$, $\underset{a}{j}{}^{\kappa}(C; s_0 \| s)$ is one in $R_m(C; s_0 \| s)$

* This condition, under the name of 'covariant stationär', has already been used freely in Struik (**9**) and Schouten-Struik (**8**)—see, for example, the latter work II, 13, (2.12), and 85, (10.32)—but no explicit statement of its characteristic property as stated in our definition of auto-parallel $R_m(s)$ seemed to have been made there.

(cf. Theorem 2.3). Since, by continuity, each of $\underset{ab}{\epsilon}$, $\underset{a}{\epsilon}$, ϵ defined for any m-Bein along (C) has the same value at every point of (C), and $\underset{a}{j}^{\kappa}(s)$ and $\underset{a}{j}^{\kappa}(C; s_0 \| s)$ coincide at $P(s_0)$, the $\underset{ab}{\epsilon}$, $\underset{a}{\epsilon}$, ϵ defined for $\underset{a}{j}^{\kappa}(s)$ are identical with those for $\underset{a}{j}^{\kappa}(C; s_0 \| s)$ at every point of (C).

Forming the m-vectors $j^{\kappa_1 \cdots \kappa_m}(s)$ and $j^{\kappa_1 \cdots \kappa_m}(C; s_0 \| s)$, we shall now define the *cosine of the angle between* $R_m(s)$ *and* $R_m(C; s_0 \| s)$ *at* $P(s)$ by the equation

$$\cos\theta(s, s_0) = \epsilon m! j^{\kappa_1 \cdots \kappa_m}(s) j_{\kappa_1 \cdots \kappa_m}(C; s_0 \| s). \tag{3.1}$$

We justify this definition by proving the theorem

THEOREM 3.1. *The scalar function of* s *and* s_0 *on the right-hand side of* (3.1) *has the following properties:*

(i) *it depends only on* $R_m(s)$ *and* s_0 *but not on the choice of the* m*-Bein* $\underset{a}{j}^{\kappa}(s)$ *in* $R_m(s)$*;*

(ii) *as* $P(s)$ *approaches* $P(s_0)$*, then* $R_m(s)$ *and* $R_m(C; s_0 \| s)$ *approach each other, and the scalar function approaches unity;*

(iii) *it reduces to the cosine of the angle between two vectors when* $m = 1$.

To prove (i), let $\underset{a}{'j}^{\kappa}(s)$ be another m-Bein in $R_m(s)$ and let $\underset{ab}{'\epsilon}$, $\underset{a}{'\epsilon}$, $'\epsilon$ and $\underset{a}{'j}^{\kappa}(C; s_0 \| s)$ be similarly defined. Then we have

$$'j^{\kappa_1 \cdots \kappa_m}(C; s_0 \| s_0) = 'j^{\kappa_1 \cdots \kappa_m}(s_0). \tag{2.8'}$$

From (2.3) and (2.5) it follows that

$$'j^{\kappa_1 \cdots \kappa_m}(s) = \Delta(s) . j^{\kappa_1 \cdots \kappa_m}(s), \tag{3.2}$$

where $\Delta(s) = \pm 1$ is the determinant of the transformation $T(s)$ in $R_m(s)$ which carries $\underset{a}{j}^{\kappa}(s)$ into $\underset{a}{'j}^{\kappa}(s)$. Similarly,

$$'j^{\kappa_1 \cdots \kappa_m}(C; s_0 \| s) = \overline{\Delta}(s) j^{\kappa_1 \cdots \kappa_m}(C; s_0 \| s), \tag{3.3}$$

and $\overline{\Delta}(s) = \pm 1$ is the determinant of the transformation $\overline{T}(s)$ in $R_m(C; s_0 \| s)$ which carries $\underset{a}{j}^{\kappa}(C; s_0 \| s)$ into $'j^{\kappa}(C; s_0 \| s)$. Since, by reference to (2.8) and (2.8'), $T(s_0)$ and $\overline{T}(s_0)$ are identical,

$$\Delta(s_0) = \overline{\Delta}(s_0).$$

Thus, by continuity,

$$\Delta(s) = \overline{\Delta}(s) = \pm 1, \tag{3.4}$$

and (i) now follows from (2.4), (3.2), (3.3), and (3.4).

We deduce (ii) from (2.8) and the following equalities:

$$\epsilon m! \overset{\kappa_1 \dots \kappa_m}{j}(s_0) \overset{}{j}_{\kappa_1 \dots \kappa_m}(s_0) = \frac{\epsilon}{m!} e^{a_1 \dots a_m} e_{b_1 \dots b_m} \Big(\overset{\kappa_1}{j}_{a_1} \dots \overset{\kappa_m}{j}_{a_m} \overset{b_1}{j}_{\kappa_1} \dots \overset{b_m}{j}_{\kappa_m} \Big)_0$$

$$= \frac{\epsilon}{m!} \delta^{a_1 \dots a_m}_{b_1 \dots b_m} \epsilon_{a_1 b_1} \dots \epsilon_{a_m b_m} = \frac{\epsilon}{m!} \sum_{a_1 \dots a_m} \delta^{a_1 \dots a_m}_{a_1 \dots a_m} \epsilon_{a_1} \dots \epsilon_{a_m} = \frac{\epsilon}{m!} \delta^{a_1 \dots a_m}_{a_1 \dots a_m} \epsilon = 1.$$

Finally, for $m = 1$, the scalar function becomes

$$\epsilon \overset{1}{j}{}^{\kappa}(s) \overset{1}{j}_{\kappa}(C; s_0 \| s)$$

which, according to Schouten's definition,* is the cosine of the angle between the vectors $\overset{1}{j}{}^{\kappa}(s)$ and $\overset{1}{j}{}^{\kappa}(C; s_0 \| s)$.

Since $\cos\theta$ as defined by (3.1) is real and positive but not necessarily less than 1, θ determined from

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

is of the form $\qquad 2k\pi \pm \alpha \quad (\tfrac{1}{2}\pi \geqslant \alpha \geqslant 0),$

or $\qquad 2k\pi \pm i\beta \quad (\beta > 0)$

according as $\cos\theta \leqslant 1$ or > 1. We shall define $\pm\alpha$ or $\pm i\beta$ as *the angle θ between $R_m(s)$ and $R_m(C; s_0 \| s)$* according as $\cos\theta \leqslant 1$ or > 1, so that when $\cos\theta \to 1$, $\theta \to 0$ and vice versa, and thus the equation

$$\lim_{\cos\theta \to 1} \frac{1 - \cos\theta}{\theta^2} = \frac{1}{2} \tag{3.5}$$

holds.

We now return to equation (3.1), defining $\cos\theta(s, s_0)$ and obtain a generalization of a formula of Hlavatý's. If we expand $\cos\theta(s, s_0)$ into a Taylor series near $P(s_0)$

$$\cos\theta(s, s_0) = \cos\theta(s_0, s_0) + (s - s_0)\{\delta_s \cos\theta(s, s_0)\}_0 + \tfrac{1}{2}(s - s_0)^2\{\delta_s^{(2)} \cos\theta(s, s_0)\}_0 + \dots$$

and make use of (3.1), $\cos\theta(s_0, s_0) = 1$ (cf. Theorem 3.1, (i)) and

$$\delta_s \overset{\kappa_1 \dots \kappa_m}{j}(C; s_0 \| s) = 0,$$

which represents the fact that $R_m(C; s_0 \| s)$ is auto-parallel, we have

$$\cos\theta(s, s_0) = 1 + (s - s_0)\epsilon m! \{\overset{}{j}_{\kappa_1 \dots \kappa_m}(C; s_0 \| s) \delta_s \overset{\kappa_1 \dots \kappa_m}{j}(s)\}_0 + \tfrac{1}{2}(s - s_0)^2 \epsilon m! \{\overset{}{j}_{\kappa_1 \dots \kappa_m}(C; s_0 \| s) \delta_s^{(2)} \overset{\kappa_1 \dots \kappa_m}{j}(s)\}_0 + \dots . \tag{3.6}$$

* Schouten-Struik (8), I. 50.

In virtue of (2.8) and

$$j_{\kappa_1\ldots\kappa_m}(s)\,\delta_s j^{\kappa_1\ldots\kappa_m}(s) = 0,$$

$$j_{\kappa_1\ldots\kappa_m}(s)\,\delta_s^{(2)} j^{\kappa_1\ldots\kappa_m}(s) = -\{\delta_s j^{\kappa_1\ldots\kappa_m}(s)\}\{\delta_s j_{\kappa_1\ldots\kappa}(s)\},$$

which are obtained by covariant differentiation of

$$j_{\kappa_1\ldots\kappa_m}(s) j^{\kappa_1\ldots\kappa_m}(s) = \frac{\epsilon}{m!},$$

equation (3.6) can be written

$$\cos\theta(s, s_0) = 1 - \tfrac{1}{2}(s-s_0)^2 \epsilon m!\,\{\delta_s j^{\kappa_1\ldots\kappa_m}(s)\}_0\{\delta_s j_{\kappa_1\ldots\kappa_m}(s)\}_0 + \ldots.$$

From this and (3.5) it follows that

$$\lim_{s\to s_0}\left\{\frac{\theta(s, s_0)}{s-s_0}\right\}^2 = \epsilon m!\,\{\delta_s j^{\kappa_1\ldots\kappa_m}(s)\}_0\{\delta_s j_{\kappa_1\ldots\kappa_m}(s)\}_0.$$

Putting $ds = \lim_{s\to s_0}(s-s_0)$, and dropping the zero index, we have

$$\boxed{\left(\frac{d\Theta}{ds}\right)^2 \equiv \left\{\frac{\theta(s+ds, s)}{ds}\right\}^2 = \epsilon m!\,(\delta_s j^{\kappa_1\ldots\kappa_m})(\delta_s j_{\kappa_1\ldots\kappa_m})}. \qquad (3.7)$$

This is the formula we wished to establish. For the particular case when $m = 1$ and V_n is ordinary, it reduces to a formula of Hlavatý's.*

The function Θ of s, defined by (3.7) to within a sign and a constant of integration, has the property that its differential $d\Theta = (d\Theta/ds)\,ds$ is equal to the angle between $R_m(s+ds)$ and $R_m(C; s \,\|\, s+ds)$. We shall call Θ the *characteristic angle of* $R_m(s)$ *along* (C) for the reason that *in an ordinary* V_n $d\Theta/ds = 0$ if and only if $\delta_s j^{\kappa_1\ldots\kappa_m} = 0$,† i.e. Θ *is constant if and only if* $R_m(s)$ *is auto-parallel along* (C). We remark that although the characteristic angle Θ may not be real, in what follows we shall only be concerned with the real quantity $(d\Theta/ds)^2$.

* Hlavatý (**5**), 11, formula (13).

† This follows from the fact that, if, in an ordinary V_n, an affinor $v^{\kappa_1\ldots\kappa_m}$ satisfies the equation

$$v^{\kappa_1\ldots\kappa_m} v_{\kappa_1\ldots\kappa_m} \equiv a_{\kappa_1\lambda_1}\ldots a_{\kappa_m\lambda_m} v^{\kappa_1\ldots\kappa_m} v^{\lambda_1\ldots\lambda_m} = 0, \qquad (a)$$

then $v^{\kappa_1\ldots\kappa_m} = 0$. To prove this, let P be an arbitrary but fixed point. Then we can always find a system of real coordinates $\xi^{\kappa'}$ for which $a^{\kappa'\lambda'} = \delta^{\lambda'}_{\kappa'}$ at P (cf. Eisenhart (**4**), 23). In $\xi^{\kappa'}$ and at P, (a) gives

$$\sum_{\kappa'_1\ldots\kappa'_m} (v^{\kappa'_1\ldots\kappa'_m})^2 = 0,$$

which shows that $v^{\kappa'_1\ldots\kappa'_m}$ and consequently also $v^{\kappa_1\ldots\kappa_m}$ vanish at P.

4. Frenet formulae for a V_m in a V_n†

Let E_{m_0} $(m_0 = m)$ be the tangent space and $E_{m_0+m_1+\ldots+m_x}$ the xth osculating space (or osculating space of order x) of a V_m in a V_n at any point P, where $x = 1, 2, \ldots$ and $m_0+m_1+\ldots+m_x \leqslant n$ is the number of dimensions. Then, since, at P, $E_{m_0+m_1+\ldots+m_x}$ contains E_{m_0}, $E_{m_0+m_1}, \ldots, E_{m_0+m_1+\ldots+m_{x-1}}$, and all the osculating spaces are contained in the local R_n of V_n at P, there exists at every point of V_m a smallest integer k such that the osculating spaces at that point of orders not less than k all coincide. If none of the osculating spaces is tangent to the null-cone of V_n, then, analogous to the principal normals of a curve, we have 'normal spaces' R_{m_x} $(x = 1, \ldots, k)$ of different orders, the xth R_{m_x} being the linear space of highest dimensions contained in $E_{m_0+m_1+\ldots+m_x}$ and orthogonal to $E_{m_0+m_1+\ldots+m_{x-1}}$. For completeness we put $R_{m_0} = E_{m_0}$.

The existence of the normal spaces, which we shall assume for the rest of this paper, permits us to introduce in R_{m_x} an m_x-Bein

$$\left.\begin{array}{l} \overset{p_x}{i}{}^{\kappa}, \\ p_x, q_x, r_x, t_x = m_0+m_1+\ldots+m_{x-1}+1, \ldots, m_0+m_1+\ldots+m_x, \\ p_0, q_0, r_0, t_0 = 1, \ldots, m, \end{array}\right\} \quad (4.1)$$

with $$\underset{p_x}{\epsilon} = \underset{p_x}{i^{\kappa}}\, \overset{p_x}{i}_{\kappa} \quad (p_x \text{ not summed}), \qquad \underset{(x)}{\epsilon} = \text{product of } \underset{p_x}{\epsilon}, \quad (4.2)$$

where $\underset{(x)}{\epsilon}$ is the signum of R_{m_x}.

In terms of $\underset{p_x}{i^{\kappa}}$, the connecting affinor $\overset{x}{B}{}^{\kappa}_{p_x}$ and the induced fundamental tensor $\overset{x}{a}_{q_x p_x}$ of R_{m_x} are given by

$$\overset{x}{B}{}^{\kappa}_{p_x} = \underset{q_x}{i^{\kappa}}\, \overset{q_x}{i}_{p_x}, \qquad \overset{x}{a}_{q_x p_x} = \overset{x}{B}{}^{\lambda}_{q_x}\, \overset{x}{B}{}^{\kappa}_{p_x} a_{\lambda\kappa} \quad (x = 0, 1, \ldots, k). \quad (4.3)$$

We note that $\overset{0}{B}{}^{\kappa}_{p_x}$ and $\overset{0}{a}_{q_x p_x}$ are respectively the connecting affinor and the first fundamental tensor of V_m.

* We denote by V_m $(m < n)$ an m-dimensional subspace X_m (or a domain of it) of V_n whose tangent space is not tangent to the null-cone of V_n. If V_n is ordinary, this condition is automatically satisfied by every X_m. The same remark applies to the conditions required for the establishment of the Frenet formulae for a V_m in V_n. All the formulae in this section are taken from Schouten-Struik (**8**), II § 13, except that a few modifications in the use of the indices have been made: $p_x, q_x, \ldots$ in (4.1) and u, v in (4.4).

If we consider the V_m, not as a single V_m, but as a member of ∞^{n-m} V_m's so that there is always one and only one of these V_m's passing through each point of V_n, then the D-symbols of van der Waerden and Bortolotti can be applied to this case and we have, among others, the Frenet formulae for the V_m in V_n

$$\left.\begin{aligned} D_{r_0}\overset{x}{B}{}^{\kappa}_{q_x} &= -\overset{x}{H}{}^{\kappa}_{r_0 . q_x} + \overset{x+1}{H}{}^{\kappa}_{r_0 q_x} \\ \overset{0}{H}{}^{\kappa}_{r_0 . q_0} &= 0, \qquad \overset{k+1}{H}{}^{\kappa}_{r_0 q_k} = 0 \end{aligned}\right\} \quad (x = 0, 1, \ldots, k), \tag{4.4}$$

where $\overset{x+1}{H}{}^{\kappa}_{r_0 q_x}$ with the indices r_0, q_x, κ lying respectively in R_{m_0}, R_{m_x}, $R_{m_{x+1}}$ is called the '$(x+1)$th curvature affinor of V_m with respect to V_n'. We remark that the affinors $\overset{x}{B}{}^{\kappa}_{p_x}$, $\overset{x}{a}_{q_x p_x}$, $\overset{x+1}{H}{}^{\kappa}_{r_0 q_x}$ are independent of the choice of the m_x-Beins $\overset{p_x}{i}{}^{\kappa}$ in R_{m_x} and that the Frenet formulae (4.4) hold in a domain of V_m throughout which the integers k, m_1, $m_2, \ldots, m_\kappa$ remain of the same values.

If we define the vectors $\overset{u}{\underset{v}{v}}_{r_0}$ in V_m by

$$\overset{u}{\underset{v}{v}}_{r_0} = \overset{u}{\underset{v}{i}}_{\kappa} D_{r_0} i^{\kappa} = -i^{\kappa} D_{r_0} \overset{u}{\underset{v}{i}}_{\kappa} = \overset{0}{B}{}^{\mu}_{r_0} \overset{u}{\underset{v}{i}}_{\kappa} \nabla_{\mu} i^{\kappa} = \pm \overset{v}{\underset{u}{v}}_{r_0}$$
$$(u, v = 1, 2, \ldots, m_0+m_1+\ldots+m_k), \tag{4.5}$$

then

$$\overset{x+1}{H}{}^{\kappa}_{r_0 q_x} = \overset{t_{x+1}}{\underset{p_x}{v}}_{r_0} \overset{p_x}{\underset{t_{x+1}}{i}}{}^{\kappa} \overset{p_x}{i}_{q_x} \quad (x = 1, \ldots, k),$$

$$\overset{x+1}{H}{}^{\kappa}_{r_0 . q_{x+1}} = -\overset{p_x \; t_{x+1}}{\underset{t_{x+1}}{v}}_{r_0} \overset{p_x}{i}_{q_{x+1}} \overset{i^{\kappa}}{\underset{p_x}{}} \quad (x = 1, \ldots, k-1), \tag{4.6}$$

and the Frenet formulae (4.4) with the exception of the first one can be written

$$D_{r_0} \overset{\kappa}{i}_{q_x} = \overset{t_{x+1}}{\underset{q_x \; t_{x+1}}{v}}_{r_0} \overset{\kappa}{i} + \overset{t_x}{\underset{q_x \; t_x}{v}}_{r_0} \overset{\kappa}{i} + \overset{t_{x-1}}{\underset{q_x \; t_{x-1}}{v}} \overset{\kappa}{i} \quad (x = 1, \ldots, k), \tag{4.7}$$

where it is to be noted that $\overset{t_{k+1}}{\underset{q_k}{v}}_{r_0} = 0$.

5. Geometric interpretation of the affinors $\overset{x+1}{H}{}^{\kappa}_{r_0 \lambda}$ $\overset{x+1}{H}{}^{\lambda}_{p_0 . \kappa}$

Let i^{κ} be the tangent vector of a curve (C) in V_m. At each point of (C) there is defined an $R_{m'}$ ($m' = m_y+m_{y+1}+\ldots+m_{y+z}$; $0 \leqslant y, z$; $y+z \leqslant k$),* which is composed of the yth, $(y+1)$th,..., $(y+z)$th

* By taking all the possible values of the integers y and z satisfying these two inequalities, we have $\frac{1}{2}(k+1)(k+2)$ different $R_{m'}$'s.

normal spaces R_{m_y}, $R_{m_{y+1}}$,..., $R_{m_{y+z}}$ of $V_{m'}$ and therefore spanned by the m'-Bein $\underset{p_y}{i^\kappa}$, $\underset{p_{y+1}}{i^\kappa}$,..., $\underset{p_{y+z}}{i^\kappa}$. Omitting for simplicity most of the indices we suppose that in the m'-vector

$$i^{\kappa_1 \ldots \kappa_{m'}} = \underbrace{i^{[\kappa_1} \ldots i}_{m_y} \; \underbrace{i \; \ldots \; i}_{m_{y+1}} \ldots \underbrace{i \; \ldots \; i^{\kappa_{m'}]}}_{m_{y+z}}, \tag{5.1}$$

which represents the $R_{m'}$, the i^κ's are arranged in the natural order of their subscripts* so that the first m_y are the vectors $\underset{p_y}{i^\kappa}$, the next m_{y+1} are the vectors $\underset{p_{y+1}}{i^\kappa}$, etc.

Let $\Theta_{(yz)}$ be the characteristic angle of $R_{m'}$ along (C), and note that $\delta_s = i^{r_0} D_{r_0}$, then we have from (3.7)

$$\left\{\frac{d\Theta_{(yz)}}{ds}\right\}^2 = \underset{(y)}{\epsilon} \; \underset{(y+1)}{\epsilon} \ldots \underset{(y+z)}{\epsilon} (m')! \, (i^{r_0} D_{r_0} i^{\kappa_1 \ldots \kappa_{m'}})(i^{p_0} D_{p_0} i_{\kappa_1 \ldots \kappa_{m'}}), \tag{5.2}$$

where $\underset{(y)}{\epsilon}$, $\underset{(y+1)}{\epsilon}$,..., $\underset{(y+z)}{\epsilon}$ are respectively the signa of R_{m_y}, $R_{m_{y+1}}$,..., $R_{m_{y+z}}$. Differentiating $i^{\kappa_1 \ldots \kappa_{m'}}$ covariantly along (C), we get

$$\begin{aligned} i^{r_0} D_{r_0} i^{\kappa_1 \ldots \kappa_{m'}} = {} & \sum_{q_y} \underbrace{i^{[\kappa_1} \ldots \left(i^{r_0} D_{r_0} \underset{q_y}{i} \right) \ldots i}_{m_y} \; \underbrace{i \; \ldots \; i}_{m_{y+1}} \ldots \underbrace{i \; \ldots \; i^{\kappa_{m'}]}}_{m_{y+z}} + \\ & + \sum_{q_{y+1}} \underbrace{i^{[\kappa_1} \ldots i}_{m_y} \; \underbrace{i \ldots \left(i^{r_0} D_{r_0} \underset{q_{y+1}}{i} \right) \ldots i}_{m_{y+1}} \ldots \underbrace{i \; \ldots \; i^{\kappa_{m'}]}}_{m_{y+z}} + \\ & + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad + \\ & + \sum_{q_{y+z}} \underbrace{i^{[\kappa_1} \ldots i}_{m_y} \; \underbrace{i \; \ldots \; i}_{m_{y+1}} \ldots \underbrace{i \ldots \left(i^{r_0} D_{r_0} \underset{q_{y+z}}{i} \right) \ldots i^{\kappa_{m'}]}}_{m_{y+z}}. \end{aligned} \tag{5.3}$$

From the Frenet formulae (4.7) and the equation

$$i^{r_0} D_{r_0} \underset{q_0}{i^\kappa} = \overset{0}{B}{}^\kappa_{p_0} \, i^{r_0} \nabla'_{r_0} \underset{q_0}{i^{p_0}} + i^{r_0} \underset{q_0}{i^{p_0}} \overset{1}{H}{}_{r_0 p_0}^{\cdot \; \cdot \; \kappa}, \dagger \tag{5.4}$$

which represents the well-known decomposition of the *associate* curvature vector *on the left* into its tangential and normal components, it is seen that only the first and the last sums of (5.3)

* Cf. (4.1).

† Equivalent to Schouten-Struik (8), II 85 (10.5α).

can survive. In fact, we have according to the values of y and z the following three cases:

$$(y \neq 0) \qquad i^{r_0} D_{r_0} i^{\kappa_1 \ldots \kappa_{m'}} = \sum_{q_y} i^{r_0} v_{\substack{r_0 \\ q_y}}^{t_y - 1} \underbrace{i^{[\kappa_1} \ldots \underbrace{i}_{t_{y-1}} \ldots i}_{m_y} \underbrace{i \ldots i}_{m_{y+1}} \underbrace{i \ldots i \ldots i}_{m_{y+z}} i^{\kappa_{m'}]} +$$

$$+ \sum_{q_{y+z}} i^{r_0} v_{\substack{r_0 \\ q_{y+z}}}^{t_{y+z+1}} \underbrace{i^{[\kappa_1} \ldots i}_{m_y} \underbrace{i \ldots i}_{m_{y+1}} \underbrace{i \ldots \underbrace{i}_{t_{y+z+1}} \ldots i}_{m_{y+z}} i^{\kappa_{m'}]}; \quad (5.5)$$

$$(y = 0;\ z \neq 0) \qquad i^{r_0} D_{r_0} i^{\kappa_1 \ldots \kappa_{m'}} = \sum_{q_z} i^{r_0} v_{\substack{r_0 \\ q_z}}^{t_{z+1}} \underbrace{i^{[\kappa_1} \ldots i}_{m_0} \underbrace{i \ldots i}_{m_1} \underbrace{i \ldots i \ldots \underbrace{i}_{t_{z+1}} \ldots i}_{m_z} i^{\kappa_{m'}]}; \quad (5.5')$$

$$(y = z = 0) \qquad i^{r_0} D_{r_0} i^{\kappa_1 \ldots \kappa_m} = \sum_{q_0} \underbrace{i^{[\kappa_0} \ldots \left(i^{r_0} i^{p_0} \overset{1}{H}_{r_0 \cdot p_0}^{\cdot \cdot \kappa_0} \right) \ldots i^{\kappa_m]}}_{m}. \quad (5.5'')$$

Taking account of (4.2) and the fact that $i^{\kappa}_{p_y}, i^{\kappa}_{p_{y+1}}, \ldots, i^{\kappa}_{p_{y+z}}$ form an m'-Bein, we obtain from (5.5)

$$(i^{r_0} D_{r_0} i^{\kappa_1 \ldots \kappa_{m'}})(i^{p_0} D_{p_0} i_{\kappa_1 \ldots \kappa_{m'}}) = \frac{1}{(m')!} \epsilon_{(y)}^{\ \ \ldots} \epsilon_{(y+z)} \times$$

$$\times \Big\{ \sum_{t_{y-1}} \sum_{q_y} \epsilon_{t_{y-1}} \epsilon_{q_y} \left(i^{r_0} v_{\substack{r_0 \\ q_y}}^{t_{y-1}} \right)^2 + \sum_{q_{y+z}} \sum_{t_{y+z+1}} \epsilon_{q_{y+z}} \epsilon_{t_{y+z+1}} \left(i^{r_0} v_{\substack{r_0 \\ q_{y+z}}}^{t_{y+z+1}} \right)^2 \Big\}. \quad (5.6)$$

To express the right-hand member of this equation in terms of the affinor H's, we make use of (4.6), from which we have

$$i^{r_0} i^{p_0} \overset{x+1}{H}_{r_0 \lambda}^{\cdot \cdot \kappa} \overset{x+1}{H}_{p_0 \cdot \kappa}^{\cdot \cdot \lambda} = \sum_{q_x} \sum_{t_{x+1}} \epsilon_{q_x} \epsilon_{t_{x+1}} \left(i^{r_0} v_{\substack{r_0 \\ q_x}}^{t_{x+1}} \right)^2 = \sum_{q_x} \sum_{t_{x+1}} \epsilon_{q_x} \epsilon_{t_{x+1}} \left(i^{r_0} v_{t_{x+1}}^{q_x} \right)^2. \quad (5.7)$$

Hence from (3.2), (3.6), and (3.7) follow our final formulae which it is the purpose of this note to establish

$$\boxed{\left(\frac{d\Theta_{(yz)}}{ds} \right)^2 = \left(\overset{y}{H}_{r_0 \lambda}^{\cdot \cdot \kappa} \overset{y}{H}_{p_0 \cdot \kappa}^{\cdot \cdot \lambda} + \overset{y+z+1}{H}_{r_0 \lambda}^{\cdot \cdot \kappa} \overset{y+z+1}{H}_{p_0 \cdot \kappa}^{\cdot \cdot \lambda} \right) i^{r_0} i^{p_0}}. \quad (5.8)$$

Because $\overset{0}{H}_{r_0 \lambda}^{\cdot \cdot \kappa} = 0$, these formulae also hold for the cases when $y = 0$, as is readily verified by applying the same process to (5.5′) and (5.5″). When $y = 0$, the $R_{m'}$ becomes the zth osculating space;

when $y = z = 0$, $R_{m'}$ becomes the tangent space. For these two cases, we have respectively

$$\boxed{\left\{\frac{d\Theta_{(0z)}}{ds}\right\}^2 = i^{r_0} i^{p_0} \overset{z+1}{H}_{r_0 \dot{\lambda}}{}^{\kappa} \overset{z+1}{H}_{p_0 \cdot \kappa}{}^{\lambda}}, \tag{5.8'}$$

$$\left\{\frac{d\Theta_{(00)}}{ds}\right\}^2 = i^{r_0} i^{p_0} \overset{1}{H}_{r_0 \dot{\lambda}}{}^{\kappa} \overset{1}{H}_{p_0 \cdot \kappa}{}^{\lambda}. * \tag{5.8''}$$

Formulae (5.8′) furnish a geometric interpretation of the affinors $\overset{z+1}{H}_{r_0 \dot{\lambda}}{}^{\kappa} \overset{z+1}{H}_{p_0 \cdot \kappa}{}^{\lambda}$ of V_m.

As a consequence of (5.8) and (5.8′) we find the following interesting equation

$$\boxed{\left\{\frac{d\Theta_{(yz)}}{ds}\right\}^2 = \left\{\frac{d\Theta_{(0y-1)}}{ds}\right\}^2 + \left\{\frac{d\Theta_{(0y+z)}}{ds}\right\}^2}. \tag{5.9}$$

For a curve, $m_0 = m_1 = \ldots = 1$, and†

$$\overset{x}{H}_{\mu \dot{\lambda}}{}^{\kappa} = \underset{x}{\kappa} \overset{0}{i}_{\mu} \overset{x-1}{i}_{\lambda} i^{\kappa}, \qquad i^{\kappa} = \underset{0}{i^{\kappa}}.$$

Therefore (5.8) and (5.8′) become respectively

$$\left\{\frac{d\Theta_{(yz)}}{ds}\right\}^2 = \underset{y}{\epsilon}\left(\underset{y-1}{\epsilon}\ \underset{y}{\kappa^2} + \underset{y+z+1}{\epsilon}\ \underset{y+z+1}{\kappa^2}\right), \tag{5.10}$$

$$\left\{\frac{d\Theta_{(0z)}}{ds}\right\}^2 = \underset{0}{\epsilon}\ \underset{z+1}{\epsilon}\ \underset{z+1}{\kappa^2}. \tag{5.10'}$$

When V_n is ordinary, (5.10) with $z = 0$ and (5.10′) are identical with (1.3) and (1.2) respectively.

6. Some further consequences

(i) Since the scalar function on the right-hand side of (5.8) depends only on the direction of (C) at the point P considered and is the same for all curves tangent to (C) at that point, the same must also be true of $(d\Theta_{(yz)}/ds)^2$.

If $\overset{a}{i}{}^{r_0}$ $(a = 1, \ldots, m)$ is an m-Bein in the tangent space $R_m = E_m$ of V_m, then the fundamental tensor of V_m is

$$\underset{0}{a^{r_0 p_0}} = \sum_a \underset{a}{\epsilon}\, \overset{a}{i}{}^{r_0} \overset{a}{i}{}^{p_0}, \qquad \underset{a}{\epsilon} = \underset{a}{i^{\kappa}} \overset{a}{i}_{\kappa} = \underset{a}{i^{r_0}} \overset{a}{i}_{r_0} \quad (a \text{ not summed}).$$

* Another geometric meaning of the invariant on the right-hand side has been obtained by Davies (**1**), the last formula.

† Schouten-Struik (**8**), II 120 (13.24).

Hence it follows from (5.8) that

$$\sum_{a} \epsilon_{a} \left\{ \frac{d\Theta_{(yz)}}{ds} \right\}^2 = \overset{y}{H}_{\mu\,\dot{\lambda}}{}^{\kappa} \overset{y}{H}{}^{\mu\lambda}_{\cdot\cdot\,\kappa} + \overset{y+z+1}{H}_{\mu\,\dot{\lambda}}{}^{\kappa} \overset{y+z+1}{H}{}^{\mu\lambda}_{\cdot\cdot\,\kappa}, \tag{6.1}$$

where $\overset{a}{s}$ denotes the arc-length of a curve in V_m tangent to $\overset{a}{i}{}^{r_0}$ at P. This equation shows that the left-hand member of (6.1) is independent of the choice of the m-Bein $\overset{a}{i}{}^{r_0}$ in R_m.

(ii) Putting $z = 0$ in (5.8) we have

$$\left\{ \frac{d\Theta_{(y0)}}{ds} \right\}^2 = \left(\overset{y}{H}_{r_0\,\dot{\lambda}}{}^{\kappa} \overset{y}{H}{}^{\lambda}_{p_0\,\cdot\,\kappa} + \overset{y+1}{H}_{r_0\,\dot{\lambda}}{}^{\kappa} \overset{y+1}{H}{}^{\lambda}_{p_0\,\cdot\,\kappa} \right) i^{r_0} i^{p_0}, \tag{6.2}$$

where $\Theta_{(y0)}$ is the characteristic angle of the yth normal space R_{m_y} along (C). Let the y in (6.2) assume the values 0, 1,..., k. Adding and subtracting the results alternately and remembering that $\overset{0}{H}_{r_0\,\dot{\lambda}}{}^{\kappa} = \overset{k+1}{H}_{r_0\,\dot{\lambda}}{}^{\kappa} = 0$, we get the *further generalization of Lancret's formula**

$$\sum_{x=0}^{k} (-1)^x \left\{ \frac{d\Theta_{(x0)}}{ds} \right\}^2 = 0. \tag{6.3}$$

which holds for *any* curve in V_m.

(iii) Multiplying the right-hand side of (5.5′) by the m'-vector

$$\overbrace{i_{[\kappa_1}\ldots i}^{m_0}\ \overbrace{i\ldots i}^{m_1}\ \ldots\ \overbrace{i\ldots\ i\ \ldots i_{\kappa_{m'}]}}^{m_z}{}^{p_{z+1}}$$

and summing each of the indices $\kappa_1, \kappa_2, \ldots, \kappa_{m'}$, we see by reference to Theorem 2.4 that the necessary and sufficient conditions for the auto-parallelism of the zth osculating space of V_m along (C) are

$$i^{r_0} \overset{p_{z+1}}{v}_{r_0}{}_{q_z} = 0. \tag{6.4}$$

From (4.6), it follows, because of the independence of $\overset{p_z}{i}{}^{\kappa}$ and of $\overset{p_{z+1}}{i}{}^{\kappa}$, that these conditions are equivalent to

$$i^{r_0} \overset{z+1}{H}_{r_0 q_z}{}^{\cdot\,\kappa} = 0; \tag{6.4′}$$

or, in words: *the zth osculating space of V_m is auto-parallel along (C) if and only if the $(z+1)$th curvature affinor has no component along (C).*

* Cf. (1.4).

Since the indices p_{z+1} and q_z have m_{z+1} and m_z values respectively, (6.4) are $m_{z+1}m_z$ homogeneous linear equations in the $m_0 = m$ unknowns i^{r_0}. Therefore, when $m > m_{z+1}m_z$, there exists in V_m at least one congruence of curves along each of which the zth osculating space of V_m is auto-parallel. This sufficient condition $m > m_{z+1}m_z$ is, however, not necessary, because the $m_{z+1}m_z$ linear equations (6.4) are not necessarily independent.

In order that (6.4′) may be satisfied for all curves in V_m, we must have

$$\boxed{H^{z+1}_{r_0 q_z \cdot \cdot}{}^{\kappa} = 0}\ ,* \tag{6.5}$$

and conversely. These are the conditions necessary and sufficient for the auto-parallelism of the zth osculating space over V_m.

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* The sufficiency of the conditions (6.5) has been established, directly from the Frenet formulae (4.4), by Schouten and van Kampen. Schouten-van Kampen (7) or Schouten-Struik (8) II Aufgabe 13.1.